UNIT-1

PARTIAL FRACTION

An algebraic fraction can be broken down into simpler parts known as "partial fractions". Consider an algebraic fraction, $(3x+5)/(2x^2-5x-3)$. This expression can be split into simple form like $(2)/(x-3) – (1)/(2x+1)$.

The simpler parts $[(2)/(x-3)]-[(1)/(2x+1)]$ are known as partial fractions.

This means that the algebraic expression can be written in the form of:

$(3x+5)/(2x^2-5x-3) = ((2)/(x-3))-((1)/(2x+1))$

Note: The partial fraction decomposition only works for the proper rational expression (the degree of the numerator is less than the degree of the denominator). In case, if the rational expression is in improper rational expression (the degree of the numerator is greater than the degree of the denominator), first do the division operation to convert into proper rational expression. This can be achieved with the help of polynomial long division method.

PARTIAL FRACTION FORMULA

The procedure or the formula for finding the partial fraction is:

1. While decomposing the rational expression into the partial fraction, begin with the proper rational expression.
2. Now, factor the denominator of the rational expression into the linear factor or in the form of irreducible quadratic factors (Note: Don't factor the denominators into the complex numbers).
3. Write down the partial fraction for each factor obtained, with the variables in the numerators, say A and B.
4. To find the variable values of A and B, multiply the whole equation by the denominator.
5. Solve for the variables by substituting zero in the factor variable.
6. Finally, substitute the values of A and B in the partial fractions.

PARTIAL FRACTIONS FROM RATIONAL FUNCTIONS

Any number which can be easily represented in the form of p/q, such that p and q are integers and $q\neq0$ is known as a rational number. Similarly, we can define a rational function as the ratio of two polynomial functions P(x) and Q(x), where P and Q are polynomials in x and $Q(x)\neq0$. A rational function is known as proper if the degree of P(x) is less than the degree of Q(x); otherwise, it is known as an improper rational function. With the help of the long division process, we can reduce improper rational functions to proper rational functions. Therefore, if P(x)/Q(x) is improper, then it can be expressed as:

P(x)Q(x)=A(x)+R(x)Q(x)

Here, A(x) is a polynomial in x and R(x)/Q(x) is a proper rational function.

We know that the integration of a function f(x) is given by F(x) and it is represented by:

$\int f(x)dx = F(x) + C$

Here R.H.S. of the equation means integral of f(x) with respect to x.

PARTIAL FRACTIONS DECOMPOSITION

In order to integrate a rational function, it is reduced to a proper rational function. The method in which the integrand is expressed as the sum of simpler rational functions is known as decomposition into partial fractions. After splitting the integrand into partial fractions, it is integrated accordingly with the help of traditional integrating techniques. Here the list of Partial fractions formulas is given.

PARTIAL FRACTION OF IMPROPER FRACTION

An algebraic fraction is improper if the degree of the numerator is greater than or equal to that of the denominator. The degree is the highest power of the polynomial. Suppose, m is the degree of the denominator and n is the degree of the numerator. Then, in addition to the partial fractions arising from factors in the denominator, we must include an additional term: this additional term is a polynomial of degree n − m.

Note:

- A polynomial with zero degree is K, where K is a constant
- A polynomial of degree 1 is Px + Q
- A polynomial of degree 2 is Px^2+Qx+K

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

Type: 1

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x - 25}{(x - 3)(x - 4)}$ into partial fractions.

Solution:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{A}{x-3} + \frac{B}{x-4} \text{------------------}(1)$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$$7x - 25 = A(x - 4) + B(x - 3) \text{--------------} \quad (2)$$

$$7x - 25 = Ax - 4A + Bx - 3B$$

$$7x - 25 = Ax + Bx - 4A - 3B$$

$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$

$$-25 = -4A - 3B$$

Solving these equation we get

$$A = 4 \quad \text{and} \quad B = 3$$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$

$$28 - 25 = 0 + B(1)$$

$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$

$$21 - 25 = A(-1) + 0$$

$$-4 = -A$$

$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

2. Solve 3x+1/(x-2)(x+1) into partial fractions

$$\frac{3x+1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\Rightarrow \frac{3x+1}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\Rightarrow \quad 3x+1 = A(x+1) + B(x-2) \quad \ldots(1)$$

Putting $x = -1$ in (1) we get,

$$-3+1 = B(-3)$$

$$\Rightarrow \quad -2 = -3B \quad \Rightarrow \boxed{B = \frac{2}{3}}$$

Putting $x = 2$ in (1) we get,

$$6+1 = A(2+1)$$

$$\Rightarrow \quad 7 = 3A \Rightarrow \boxed{\frac{7}{3} = A}$$

$$\therefore \frac{3x+1}{(x-2)(x+1)} = \frac{\frac{7}{3}}{x-2} + \frac{\frac{2}{3}}{x+1}$$

$$= \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$$

3. Resolve $1/x^2$-1 into partial fractions

Solutios: $\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$

$1 = A(x+1) + B(x-1) \quad (1)$

Put $x - 1 = 0$, ⇨ $x = 1$ in equation (1)

$1 = A(1+1) + B(1-1)$ ⇨ $A = \frac{1}{2}$

Put $x + 1 = 0$, ⇨ $x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B$, ⇨ $B = \frac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Type: 2

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\dfrac{x^2 - 3x + 1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$$x^2 - 3x + 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \quad \text{(I)}$$

Putting $x - 1 = 0 \quad \Rightarrow \quad x = 1$ in (I), then

$$(1)^2 - 3(1) + 1 = B(1-2)$$
$$1 - 3 + 1 = -B$$
$$-1 = -B$$
$$\Rightarrow \quad B = 1$$

Putting $x - 2 = 0 \quad \Rightarrow \quad x = 2$ in (I), then

$$(2)^2 - 3(2) + 1 = C(2-1)^2$$
$$4 - 6 + 1 = C(1)^2$$
$$\Rightarrow \quad -1 = C$$

Now $x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1)$

Comparing the co-efficient of like powers of x on both sides, we get

$$A + C = 1$$
$$A = 1 - C$$
$$= 1 - (-1)$$
$$= 1 + 1 = 2$$
$$\Rightarrow \quad A = 2$$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Type 3:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x+3)(x^2+1)$, we get

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \qquad \text{(I)}$$

Put $x+3=0 \quad \Rightarrow \quad x=-3$ in Eq. (I), we have

$$9(-3)-7 = A((-3)^2+1) + (B(-3)+C)(-3+3)$$

$$-27-7 = 10A+0$$

$$A = -\frac{34}{10} \quad \Rightarrow \quad \boxed{A=-\frac{17}{5}}$$

$$9x-7 = A(x^2+1) + B(x^2+3x) + C(x+3)$$

Comparing the co-efficient of like powers of x on both sides

$$A+B=0$$

$$3B+C=9$$

Putting value of A in Eq. (i)

$$-\frac{17}{5}+B=0 \quad \Rightarrow \boxed{B=\frac{17}{5}}$$

From Eq. (iii)

$$C = 9-3B \;= 9-3\left(\frac{17}{4}\right)$$

$$= 9-\frac{51}{5} \quad \Rightarrow \quad \boxed{C=-\frac{6}{5}}$$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

LOGARITHMS

The logarithmic function is an inverse of the exponential function. It is defined as:

$y=\log_a x$, if and only if $x=a^y$; for $x>0$, $a>0$, and $a\neq1$.

Natural logarithmic function: The log function with base e is called natural logarithmic function and is denoted by $\log_e$.

$f(x) = \log_e x$

The questions of logarithm could be solved based on the properties, given below

Product rule: $\log_b MN = \log_b M + \log_b N$

Quotient rule: $\log_b M/N = \log_b M – \log_b N$

Power rule: $\log_b M^p = P \log_b M$

Zero Exponent Rule: $\log_a 1 = 0$

Change of Base Rule: $\log_b (x) = \ln x / \ln b$ or $\log_b (x) = \log_{10} x / \log_{10} b$

Logarithm Properties

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a b = \frac{1}{\log_b a}$$

The following can be derived from the above properties

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$\log_a a^r = r$$

$$\log_a \frac{1}{b} = -\log_a b$$

$$\log_{\frac{1}{a}} b = -\log_a b$$

$$\log_a b \log_b c = \log_a c$$

Solved Examples:

1. Express $5^3 = 125$ in logarithm form.

Solution:

$5^3 = 125$

As we know,

$a^b = c \Rightarrow \log_a c=b$

Therefore;

$\text{Log}_5 125 = 3$

2. Express $\log_{10} 1 = 0$ in exponential form.

Solution:

Given, $\log_{10} 1 = 0$

By the rule, we know;

$\log_a c=b \Rightarrow a^b = c$

Hence,

$10^0 = 1$

3. Find the log of 32 to the base 4.

Solution: $\log_4 32 = x$

$4^x = 32$

$(2^2)^x = 2x2x2x2x2$

$2^{2x} = 2^5$

$2x=5$

$x=5/2$

Therefore,

$\log_4 32 =5/2$

4. Find x if $\log_5(x-7)=1$.

Solution: Given,

$\log_5(x-7)=1$

Using logarithm rules, we can write;

$5^1 = x-7$

$5 = x-7$

$x=5+7$

$x=12$

5. If $\log_a m=n$, express a^{n-1} in terms of a and m.

Solution:

$\log_a m=n$

$a^n=m$

$a^n/a=m/a$

$a^{n-1}=m/a$

6. Solve for x if $\log(x-1)+\log(x+1)=\log_2 1$

Solution: $\log(x-1)+\log(x+1)=\log_2 1$

$\log(x-1)+\log(x+1)=0$

$\log[(x-1)(x+1)]=0$

Since, $\log 1 = 0$

$(x-1)(x+1) = 1$

$x^2-1=1$

$x^2=2$

$x=\pm \sqrt{2}$

Since, log of negative number is not defined.

Therefore, $x=\sqrt{2}$

7. Express $\log(75/16)-2\log(5/9)+\log(32/243)$ in terms of log 2 and log 3.

Solution: $\log(75/16)-2\log(5/9)+\log(32/243)$

Since, $n\log_a m=\log_a m^n$

$\Rightarrow\log(75/16)-\log(5/9)^2+\log(32/243)$

$\Rightarrow\log(75/16)-\log(25/81)+\log(32/243)$

Since, $\log_a m-\log_a n=\log_a(m/n)$

$\Rightarrow\log[(75/16)\div(25/81)]+\log(32/243)$

$\Rightarrow\log[(75/16)\times(81/25)]+\log(32/243)$

$\Rightarrow\log(243/16)+\log(32/243)$

Since, $\log_a m+\log_a n=\log_a mn$

⇒log(32/16)

⇒log2

8. Express 2logx+3logy=log a in logarithm free form.

Solution: 2logx+3logy=log a

$\log x^2 + \log y^3 = \log a$

$\log x^2 y^3 = \log a$

$x^2 y^3 = \log a$

9. Prove that: 2log(15/18)-log(25/162)+log(4/9)=log2

Solution: 2log(15/18)-log(25/162)+log(4/9)=log2

Taking L.H.S.:

⇒2log(15/18)-log(25/162)+log(4/9)

⇒$\log(15/18)^2$-log(25/162)+log(4/9)

⇒log(225/324)-log(25/162)+log(4/9)

⇒log[(225/324)(4/9)]-log(25/162)

⇒log[(225/324)(4/9)]/(25/162)

⇒log(72/36)

⇒log2 (R.H.S)

10. Express $\log_{10}(2+1)$ in the form of $\log_{10}x$.

Solution: $\log_{10}(2+1)$

$=\log_{10}2+\log_{10}1$

$=\log_{10}(2\times10)$

$=\log_{10}20$

11. Find the value of x, if $\log_{10}(x-10)=1$.

Solution: Given, $\log_{10}(x-10)=1$.

$\log_{10}(x-10) = \log_{10}10$

x-10 = 10

x=10+10

x=20

12. Find the value of x, if log(x+5)+log(x-5)=4log2+2log3

Solution: Given,

$\log(x+5)+\log(x-5)=4\log2+2\log3$

$\log(x+5)(x-5) = 4\log2+2\log3$ [log mn=log m+log n]

$\log(x^2-25) = \log2^4+\log3^2$

$\log(x^2-25) = \log16+\log9$

$\log(x^2-25)=\log(16\times9)$

$\log(x^2-25)=\log144$

$x^2-25=144$

$x^2=169$

$x=\pm\sqrt{169}$

$x=\pm13$

13. Solve for x, if log(225/log15) = log x

Solution: $\log x = \log(225/\log15)$

$\log x=\log[(15\times15)]/\log15$

$\log x = \log 15^2/\log 15$

$\log x = 2\log 15/\log 15$

$\log x = 2$

Or

$\log_{10}x=2$

$10^2=x$

$x=10\times10$

$x=100$

Solved Examples

Evaluate the expression below using Log Rules.

$$\log_3 162 - \log_3 2$$

(i)Sol:

$$\log_3 162 - \log_3 2 = \log_3\left(\frac{162}{2}\right)$$
$$= \log_3(81)$$
$$= \log_3 3^4$$
$$= 4\log_3 3$$
$$= 4(1)$$
$$\log_3 162 - \log_3 2 = 4$$

$\log_2 8 + \log_2 4$

(ii)sol:

$$\log_2 8 + \log_2 4 = \log_2 2^3 + \log_2 2^2$$
$$= 3\log_2 2 + 2\log_2 2$$
$$= 3(1) + 2(1)$$
$$= 3 + 2$$
$$\log_2 8 + \log_2 4 = 5$$

Some Important Expansions

1. $\log(1+\ x) =\ x - \ \frac{x^2}{2} +\ \frac{x^3}{3} -\ \frac{x^4}{4} +\ \ldots\ldots.$
2. $e^x =\ 1+\ x+\ \frac{x^2}{2!} +\ \frac{x^3}{3!} -\ \frac{x^4}{4!} +\ \ldots\ldots.$
3. $a^x =\ 1+\ x\log a +\ \frac{x^2}{2!}(log\ a)^2 +\ \ldots\ldots.$
4. $\sin x =\ x -\ \frac{x^3}{3!} +\ \frac{x^5}{5!}\ldots\ldots$
5. $\cos x =\ 1 -\ \frac{x^2}{2!} +\ \frac{x^4}{4!}\ldots\ldots$
6. $\tan x =\ x +\ \frac{x^3}{3} +\ \frac{2}{15}x^5 +\ \ldots\ldots$

UNIT-V

DIFFERENTIAL EQUATIONS

Definition:

A differential equation is an equation in which differential coefficients occur.

Differential equations are of two types(i) Ordinary and (ii) Partial.

An ordinary differential equation is one which a single independent variable enters, either explicity or implicity. For example,

$$\frac{dy}{dx} = 2\sin x, \frac{d^2x}{dr^2} + m^2 x = 0$$

$$x2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + y = \sin x$$

are ordinary differential equations.

Variable separable.

Suppose an equation is of the form $f(x)dx + F(y)dy = 0$.

We can directly integrate this equation and the solution is $\int f(x)dx + \int F(y)dy = c$, where c is an arbitrary constant.

Problem: Solve $\frac{dy}{dx} \quad \frac{1 \quad y^2}{1 \quad x^2}^{\frac{1}{2}} = 0$

Solution:

We have $\frac{dy}{\sqrt{1 \quad y^2}} \quad \frac{dy}{\sqrt{1 \quad x^2}} = 0.$

Integrating, $\sin^{-1}y + \sin^{-1}x = c$.

Problem: Solve tany $\frac{dy}{dx}$ = cotx.

Solution: tany $\frac{dy}{dx}$ = cotx

tany dy = cotx dx

$$\int \tan y \; dy = \int \cot x \, dx$$

log secy = log sinx + logc

log secy – log sinx = logc

$$\log \frac{\sec y}{\sin x} = \log c$$

$$\frac{\sec y}{\sin x} = c .$$

Problem: Solve tanx $\sec^2$y dy + tany $\sec^2$x dx = 0

Solution:

tanx $\sec^2$y dy = - tany $\sec^2$x dx

$$\frac{\sec^2 y}{\tan y} dy \quad \frac{\sec^2 x}{\tan x} dx$$

$$\frac{\sec^2 y}{\tan y} dy \quad \frac{\sec^2 x}{\tan x} dx$$

put t = tany put u = tanx

dt = $\sec^2$y dy du = $\sec^2$x (- dx)

log t = - log u + log c

log t + log u = log c

log (tu) = log c

tu = c

tan y tan x = c.

Problem: Solve secx dy + secy dx = 0

Solution: secx dy = - secy dx

$$\frac{dy}{\sec y} \quad \frac{dx}{\sec x}$$

$$\int \cos y \, dy = \int \cos x \, dx$$

sin y = - sin x + c

sin x + sin y = c.

Linear Equation:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no products of these occur.

The linear equation of the first order is of the form $\frac{dy}{dx} + P_y = Q$, where P and Q are functions of x only.

Problem: Solve $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$.

Solution:

Divided by $1 + x^2$

$$\frac{(1+x^2)}{(1+x^2)}\frac{dy}{dx} \quad \frac{2xy}{1 \quad x^2} \quad \frac{4x^2}{1 \quad x^2}$$

$$\frac{dy}{dx} \quad \frac{2xy}{1 \quad x^2} \quad \frac{4x^2}{1 \quad x^2}$$

This is of the form $\frac{dy}{dx} + Py = Q$.

P = $\frac{2x}{1+x^2}$ and Q = $\frac{4x^2}{1 \quad x^2}$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\frac{2x}{1 \quad x^2} dx} \qquad \frac{4x^2}{1 \quad x^2} e^{\frac{2x}{1 \quad x^2} dx} dx + c \qquad \rightarrow (1)$$

$$e^{\int P dx} = e^{\frac{2x}{1 \quad x^2} dx}$$

put $t = 1 + x^2$

$dt = 2x\, dx$

$$e^{\frac{2x}{1 \quad x^2} dx} = e^{\frac{dt}{t}}$$

$= e^{\log t}$

$= t$

$e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2$. → (2)

Using (2) in (1),

$y(1 + x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)dx + c$

$y(1 + x^2) = \int 4x^2 dx + c$

$y(1 + x^2) = \frac{4x^3}{3} + c.$

Problem: Solve $\frac{dy}{dx}$ + y sec x = tan x.

Solution:

This is of the form $\frac{dy}{dx} + Py = Q$.

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

P = sec x & Q = tan x

$y e^{\int \sec x\, dx} = \int \tan x\, e^{\int \sec x\, dx} dx + c \quad --> (1)$

$Now\, e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)}$

$= \sec x + \tan x$

(1) → $y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\, dx + c$

$= \int \tan x \sec x\, dx + \int \tan^2 x\, dx + c$

$= \sec x + \int (1 - \sec^2 x)dx$

$= \sec x + \int dx - \int \sec^2 x dx$

$= \sec x + x - \tan x + c.$

Problem: Solve $\frac{dy}{dx}$ - tan xy = -2 sin x.

Solution:

This is of the form $\frac{dy}{dx} + Py = Q$.

The solution is $ye^{\int P dx} = \int Q e^{\int P dx} dx + c$

P = - tan x & Q = -2 sin x

$$ye^{\int -\tan x\, dx} = \int -2\sin x\, e^{\int -\tan x\, dx} dx + c \qquad -->(1)$$

$$Now\, e^{-\int \tan x\, dx} = e^{-\log \sec x}$$

$$= -\sec x$$

- y sec x = $\int -2\sin x(-\sec x) dx + c$

$= \int 2\sin x \sec x\, dx + c$

$= 2 \frac{\sin x}{\cos x} dx + c$

$= 2\int \tan x\, dx + c$

- y sec x = 2 log secx + c.

Problem: Solve $\cos^2 x \frac{dy}{dx}$ + y = tan x.

Solution:

Divided by $\cos^2$x .

$$\frac{\cos^2 x}{\cos^2 x}\frac{dy}{dx} \quad \frac{y}{\cos^2 x} \quad \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$$

$P = \sec^2 x$ & $Q = \tan x \sec^2 x$

The solution is $ye^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\int \sec^2 x\, dx} = \int \tan x \sec^2 x e^{\int \sec^2 x\, dx} dx + c \qquad --> (1)$$

$$Now\, e^{\int \sec^2 x\, dx} = e^{\tan x}$$

y e^{tanx} = $\int \tan x \sec^2 x e^{\tan x} dx + c$

put t = tanx

dt = sec^2x dx

y e^t = $\int t\ e^t dt + c$

= t . e^t - e^t

= e^t (t – 1) + c

y e^{tanx} = e^{tanx} (tanx – 1) + c

Problem: Solve (1 + x^2) $\frac{dy}{dx}$ + 2xy = cosx.

Solution:

Divided by 1 + x^2

$$\frac{(1+x^2)}{(1+x^2)} \frac{dy}{dx} \quad \frac{2xy}{1 \quad x^2} \quad \frac{\cos x}{1 \quad x^2}$$

$$\frac{dy}{dx} \quad \frac{2xy}{1 \quad x^2} \quad \frac{\cos x}{1 \quad x^2}$$

This is of the form $\frac{dy}{dx} + Py = Q$.

P = $\frac{2x}{1+x^2}$ and Q = $\frac{\cos x}{1+x^2}$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\frac{2x}{1 \quad x^2} dx} \quad \frac{\cos x}{1 \quad x^2} e^{\frac{2x}{1 \quad x^2} dx} dx + c \qquad \rightarrow (1)$$

$e^{\int P dx}$ = $e^{\frac{2x}{1 \quad x^2} dx}$

put t = 1 + x^2

dt = 2x dx

$$e^{\frac{2x}{1 \quad x^2}dx} = e^{\frac{dt}{t}}$$

$$= e^{\log t}$$

$$= t$$

$$e^{\frac{2x}{1 \quad x^2}dx} = 1 + x^2. \quad \rightarrow (2)$$

Using (2) in (1),

$$y(1 + x^2) = \frac{\cos x}{1 \quad x^2}(1 + x^2)dx + c$$

$$y(1 + x^2) = \int \cos x\, dx + c$$

$$y(1 + x^2) = \sin x + c.$$

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Problem: Solve $(D^2 + 5D + 6)\, y = e^x$.

Solution:

To find the C.F. solve $(D^2 + 5D + 6)\, y = 0$.

The auxiliary equation is $m^2 + 5m + 6 = 0$.

Solving, m = -2 and -3.

C.F. = $A e^{-2x} + B e^{-3x}$.

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6} e^x$$

$$= \frac{1}{12} e^x \text{ on replacing D by 1.}$$

$$y = A e^{-2x} + B e^{-3x} + \frac{1}{12} e^x .$$

Problem: Solve $(D^2 - 2mD + m^2)\, y = e^{mx}$.

Solution:

To find the C.F. solve $(D^2 - 2mD + m^2)\, y = 0$.

The auxiliary equation is $k^2 - 2mk + m^2 = 0$.

i.e., $(k - m)^2 = 0$, ∴ k = m twice.

C.F. = $e^{mx}(A + Bx)$.

$$\text{P.I.} = \frac{1}{(k - m)^2} e^{mx}$$

$$= \frac{x^2}{2} e^{mx}$$

∴ $y = e^{mx}\left(A + Bx + \frac{x^2}{2}\right)$.

Problem: Solve $(D^2 - 3D + 2)\, y = \sin 3x$.

Solution:

To find the C.F. solve $(D^2 + 5D + 6)\, y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

Solving, m = 2 and 1.

C.F. = $A e^{2x} + B e^{x}$.

$$\text{P.I.} = \frac{\sin 3x}{D^2 - 3D + 2}$$

$$= \frac{\sin 3x}{9 \quad 3D \quad 2}$$, put $D^2 = - a^2 = -9$

$$= \frac{\sin 3x}{7 \quad 3D} \quad \frac{7 \quad 3D}{7 \quad 3D}$$

$$= \frac{7 \sin 3x - 3D(\sin 3x)}{-49 + 9D^2}$$

$$= \frac{7 \sin 3x \quad 3(3 \cos 3x)}{49 \quad 9(\quad 9)}$$

$$= \frac{7\sin 3x \quad 9\cos 3x}{49 \quad 81}$$

$$= \frac{7\sin 3x \quad 9\cos 3x}{130}$$

$$= \frac{7\sin 3x \quad 9\cos 3x}{130}$$

y = C.F. + P.I.

$$= A e^{2x} + B e^{x} \qquad \frac{7\sin 3x \quad 9\cos 3x}{130} .$$

Problem: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 5x^2$.

Solution:

$(D^2 + 2D + 3)\, y = 5x^2$

To find the C.F. solve $(D^2 + 2D + 3)\, y = 0$.

The auxiliary equation is $m^2 + 2m + 3 = 0$.

$$m = \frac{-2 \quad \sqrt{2^2 \quad 4.1.3}}{2.1}$$

$$= \frac{-2 \quad \sqrt{4 \quad 12}}{2}$$

$$= \frac{2 \quad \sqrt{\;\;} \quad 8}{2}$$

$$= \frac{-2 \pm 2i\sqrt{2}}{2}$$

$$= -1 \pm i\sqrt{2}$$

$\alpha = -1$, $\beta = \sqrt{2}$

C.F. = e^{-x} (A cos $\sqrt{2}$ x + B sin $\sqrt{2}$ x)

$$\text{P.I.} = \frac{5x^2}{D^2 + 2D + 3}$$

$$= \frac{5x^2}{3 \quad 2D \quad D^2}$$

$$= \frac{5x^2}{3\ 1 \quad \frac{2D \quad D^2}{3}}$$

$$= \frac{5}{3}\ 1 \quad \frac{2D \quad D^2}{3} \quad ^{1}\ x^2$$

$$= \frac{5}{3}\ 1 \quad \frac{2D \quad D^2}{3} \quad \frac{2D \quad D^2}{3} \quad ^{2} - .... \Big] x^2$$

$$= \frac{5}{3}\ 1 \quad \frac{2D \quad D^2}{3} \quad \frac{4D^2 \quad 4D^3 \quad D^4}{9} \quad .... \Big] x^2$$

$$= \frac{5}{3}\ 1 \quad \frac{2D \quad D^2}{3} \quad \frac{4D^2}{9} \quad x^2$$ (Neglecting Higher Powers)

$$= \frac{5}{3}\Big[x^2 \quad \frac{2D(x^2) \quad D^2(x^2)}{3} \quad \frac{4D^2(x^2)}{9}$$

$$= \frac{5}{3}\Big[x^2 \quad \frac{2(2x) \quad 2}{3} \quad \frac{4(2)}{9}$$

$$= \frac{5}{3}\Big[x^2 \quad \frac{4x \quad 2}{3} \quad \frac{8}{9}$$

$$= \frac{5}{3}\Big[x^2 \quad \frac{4x}{3} \quad \frac{2}{3} \quad \frac{8}{9}$$

$$= \frac{5}{3}\Big[x^2 \quad \frac{4x}{3} \quad \frac{2}{9}$$

y = C.F. + P.I.

$= e^{-x}$ (A cos $\sqrt{2}$ x + B sin $\sqrt{2}$ x) + $\frac{5}{3}\Big[x^2 \quad \frac{4x}{3} \quad \frac{2}{9}$.

Problem: Solve $(D^2 + 4)\, y = e^{2x} \sin 2x$.

Solution:

The auxiliary equation $m^2 + 4 = 0$.

$m^2 = -4$

$m = \sqrt{\ \ 4}$

$m = \pm 2i$.

C.F. $= e^{0x}$ (A cos2x + B sin2x)

= A cos2x + B sin2x

P.I. $= \dfrac{e^{2x} \sin 2x}{D^2 + 4}$

$= \dfrac{e^{2x} \sin 2x}{(D\ \ 2)^2\ \ 4}$, replace D by D+2

$= \dfrac{e^{2x} \sin 2x}{D^2 + 4D + 8}$

$= \dfrac{e^{2x} \sin 2x}{4\ \ 4D\ \ 8}$, replace D^2 by -4

$= \dfrac{e^{2x} \sin 2x}{4D\ \ 4} \ \dfrac{4D\ \ 4}{4D\ \ 4}$

$= \dfrac{e^{2x}[4D(\sin 2x) - 4 \sin 2x]}{16D^2 - 16} = \dfrac{e^{2x}[4D(\sin 2x)\ \ 4 \sin 2x]}{16(\ \ 4)\ \ 16}$

$= \dfrac{4e^{2x}[2\cos 2x\ \ \sin 2x]}{80}$

y = C.F. + P.I.

= A cos2x + B sin2x $\ \ \dfrac{4e^{2x}[2\cos 2x\ \ \sin 2x]}{80}$.