SRI CHANDRASEKHARENDRA SARASWATHI VISWA MAHA VIDYALAYA

(University Established under section 3 of UGC Act, 1956)

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIAL: ME7T3 FINITE ELEMENT ANALYSIS

UNIT-1

Name of the Course	:	**FINITE ELEMENT ANALYSIS (FEA)**
Name of the Unit	:	**Introduction**
Name of the Topic	:	**Historical background**

1. Aim and Objectives:

- To familiarize on historical background of FEA.
- To understand the applications of FEA in various fields.

2. Pre-Test - MCQ type:

1. CAD stand for

    (a) Computer Aided Design

    (b) Computer Assisted Design

    (c) Computer Aimed Design

    (d) None of the above

2. CAE stand for

    (a) Computer Aided Ergonomics

    (b) Computer Aided Engineering

    (c) Computer Aided Engine

    (d) None of the above

3. Prerequisites

- The students should have a basic knowledge of computer aided design.

4. Theory behind – Historical background

Basic ideas of the finite element method originated from advances in aircraft structural analysis.

- In 1941, Hrenikoff presented a solution of elasticity problems using the ‘frame work method’.

- Courant's paper, which used piecewise polynomial interpolation over triangular sub regions to model torsion problems, appeared in 1943.
- Turner derived stiffness matrices for truss, beam, and other elements and presented their findings in 1956.
- The term finite element was first coined and used by Clough in 1960.

The finite element method (FEM), sometimes referred to as finite element analysis (FEA), is a computational technique used to obtain approximate solutions of boundary value problems in engineering. Simply stated, a boundary value problem is a mathematical problem in which one or more dependent variables must satisfy a differential equation everywhere within a known domain of independent variables and satisfy specific conditions on the boundary of the domain. Boundary value problems are also sometimes called field problems. The field is the domain of interest and most often represents a physical structure. The field variables are the dependent variables of interest governed by the differential equation. The boundary conditions are the specified values of the field variables (or related variables such as derivatives) on the boundaries of the field. Depending on the type of physical problem being analyzed, the field variables may include physical displacement, temperature, heat flux, and fluid velocity to name only a few.

The finite element analysis originated as a method of stress analysis in the design of aircrafts. It started as an extension of matrix method of structural analysis. Today this method is used not only for the analysis in solid mechanics, but even in the analysis of fluid flow, heat transfer, electric and magnetic fields and many others. Civil engineers use this method extensively for the analysis of beams, space frames, plates, shells, folded plates, foundations, rock mechanics problems and seepage analysis of fluid through porous media. Both static and dynamic problems can be handled by finite element analysis. This method is used extensively for the analysis and design of ships, aircrafts, space crafts, electric motors and heat engines.

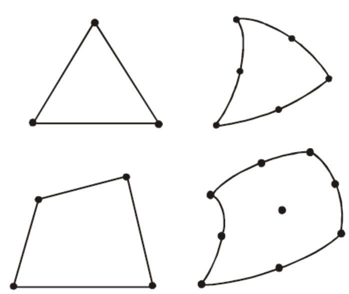
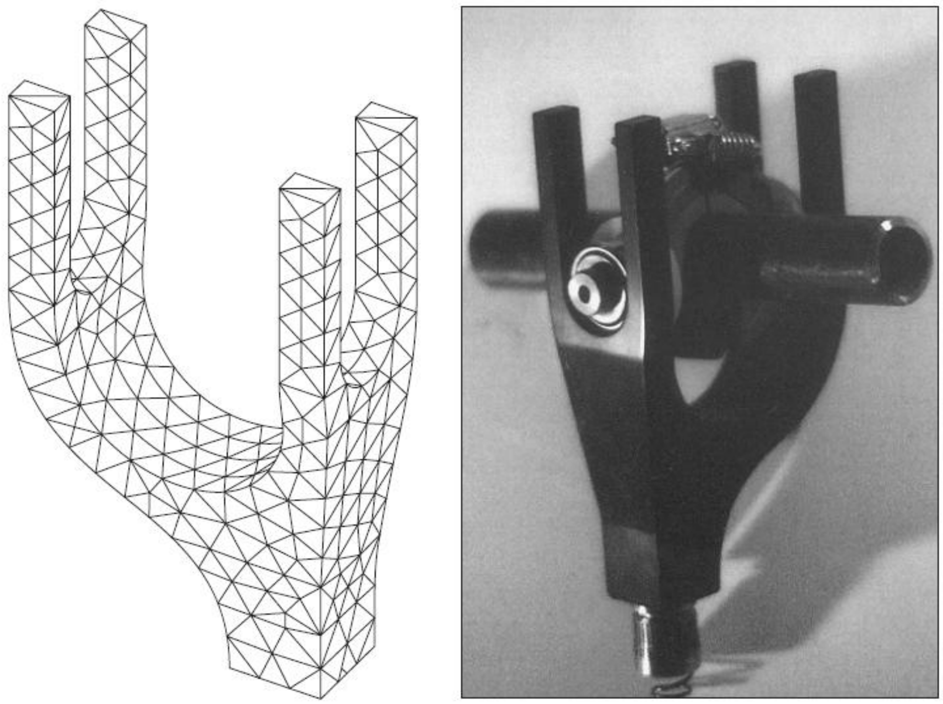


Fig 1.1 Common 2D elements

Fig 1.1 shows the common 2D elements. Figure 1.2 (a) represents the finite element model of the main load-carrying component of a prosthetic device. The device is intended to be a hand attachment to an artificial arm. In use, the hand would allow a lower arm amputee to engage in weight lifting as part of a physical fitness program. The finite element model was used to determine the stress distribution in the component in terms of the range of weight loading anticipated, so as to properly size the component and select the material.



(a) (b)

Figure 1.2.(a) A finite element model of a prosthetic hand for weightlifting.(b) Completedprototype of a prosthetic hand, attached to a bar Figure 1.2.(b) shows a prototype of the completed hand design.

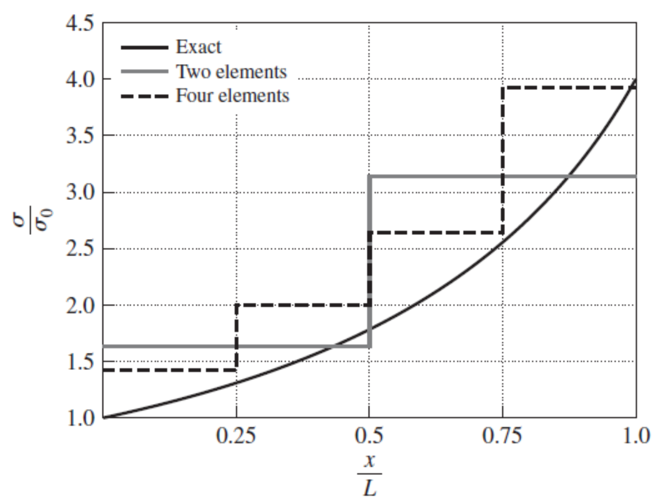


Figure 1.3. Comparison of Exact solution with Approximate solution

Examples of FEA Models

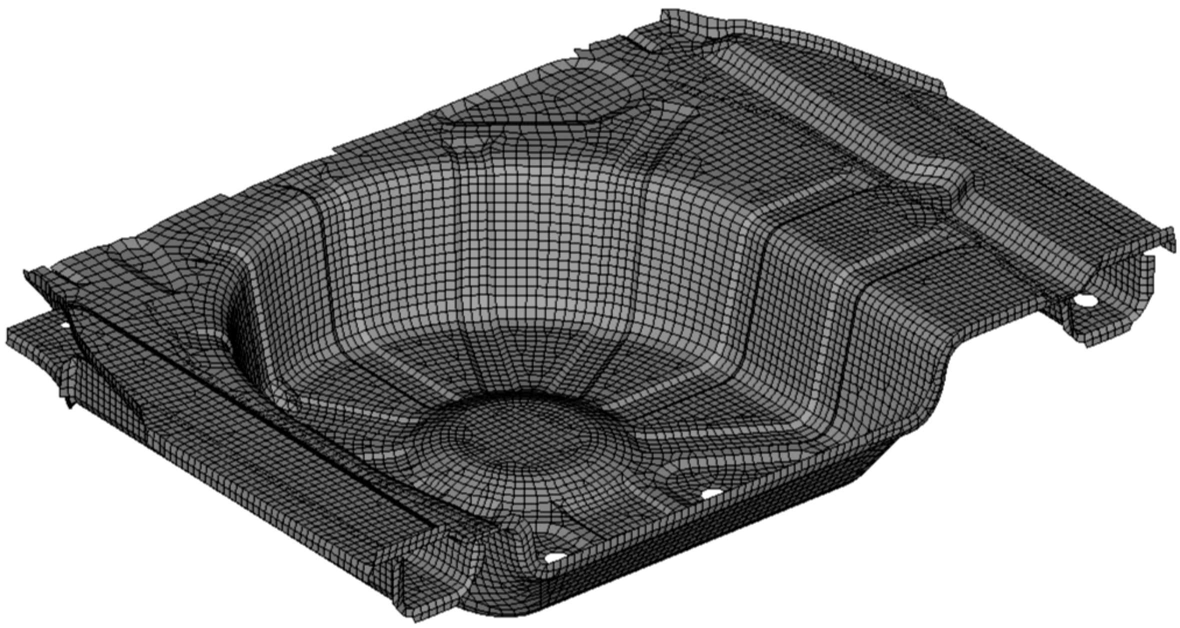


Figure 1.4. FEA Model of floor panel of an automobile

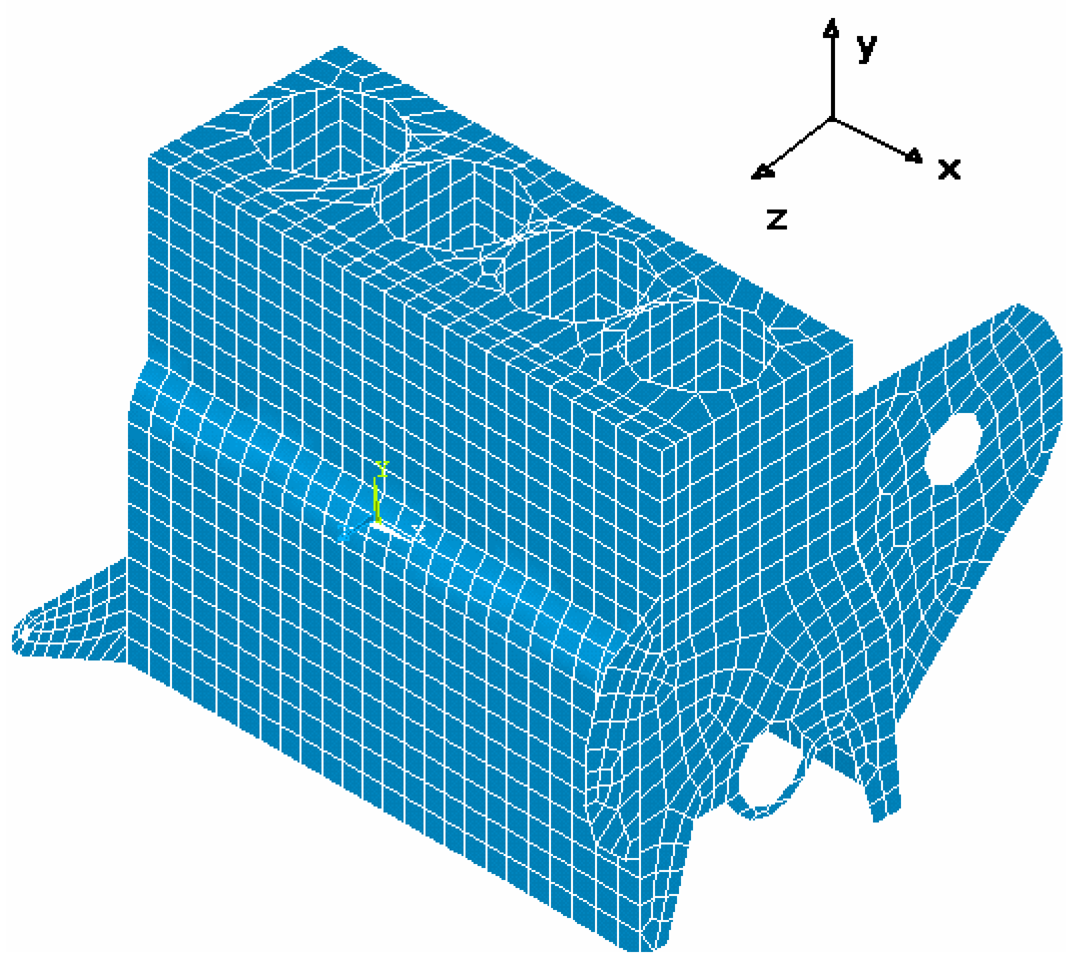


Figure 1.5. FEA Model of floor panel of an automotive engine cylinder block

5. Applications/ Simulation/ related Laboratory example

The FEA concept widely applied in all fields such as design of ships, aircrafts, space crafts, electric motors and heat engines.

6. MCQ- Post Test

1. FEA means
   (a) Finite Element Analysis
   (b) Finite Edge Analysis
   (c) Finite Extended Analysis
   (d) None of the above
2. In which year, the concept of FEA is coined
   (a) 1985
   (b) 1974
   (c) 1960
   (d) 1970
3. The Applications of FEA is applicable to
   (a) Design of ships,
   (b) Design of aircrafts
   (c) Design of Heat engines
   (d) All of the above

7. Conclusions

- The concept of FEA is applicable in all fields of applications.
- The primary advantage is to validate the with exact solutions.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=3E82-fIuSxg
https://www.youtube.com/watch?v=QbrqUjCWxb4

10. Assignments

1. Write briefly about the application of FEA in various fields.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Matrix approach**

1. Aim and Objectives:

- To apply knowledge on Matrix properties.
- To develop the matrices based on theory of elasticity

2. Pre-Test-MCQ Type

1. Deformation per unit length in the direction of force is known as
   (a) Strain
   (b) Lateral strain
   (c) Linear strain
   (d) Linear stress

2. Young's modulus is defined as the ratio of
   (a) Volumetric stress and volumetric strain
   (b) Lateral stress and lateral strain
   (c) Longitudinal stress and longitudinal strain
   (d) Shear stress to shear strain

3. Strain is defined as the ratio of
   (a) Change in volume to original volume
   (b) Change in length to original length
   (c) Change in cross-sectional area to original cross-sectional area
   (d) Any one of the above

3. Prerequisites

The basics of theory of elasticity is required.

4. Theory behind – Matrix approach

Though mathematicians, physicists and stress analysts worked independently in the field of FEM, it is the matrix displacement formulation of the stress analysts which lead to fast development of FEM. Infact till the word FEM became popular, stress analyst worked in this field in the name of matrix displacement method. In matrix displacement method, stiffness matrix of an element is assembled by direct approach while in FEM though direct stiffness matrix may be treated as an approach for assembling element properties (stiffness matrix as far as stress analysis is concerned), it is the energy approached which has revolutionized entire FEM.

The standard form of matrix displacement equation is,

$$[k]\ \{u\} = \{F\}$$

Where, $[k]$ is stiffness matrix
$\{u\}$ is displacement vector and
$\{F\}$ is force vector in the coordinate directions

The element k_{ij} of stiffness matrix maybe defined as the force at coordinate i due to unit displacement in coordinate direction j.

Example for Bar/Line Element:

Common problems in this category are the bars and columns with varying cross section subjected to axial forces as shown in Fig. 6.

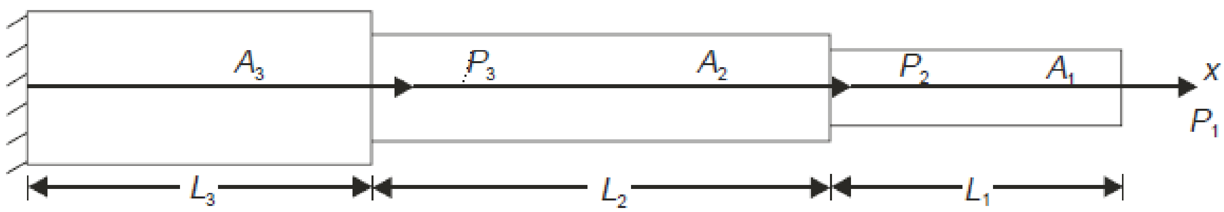


Figure 1.6. Stepped Bar subjected to axial force

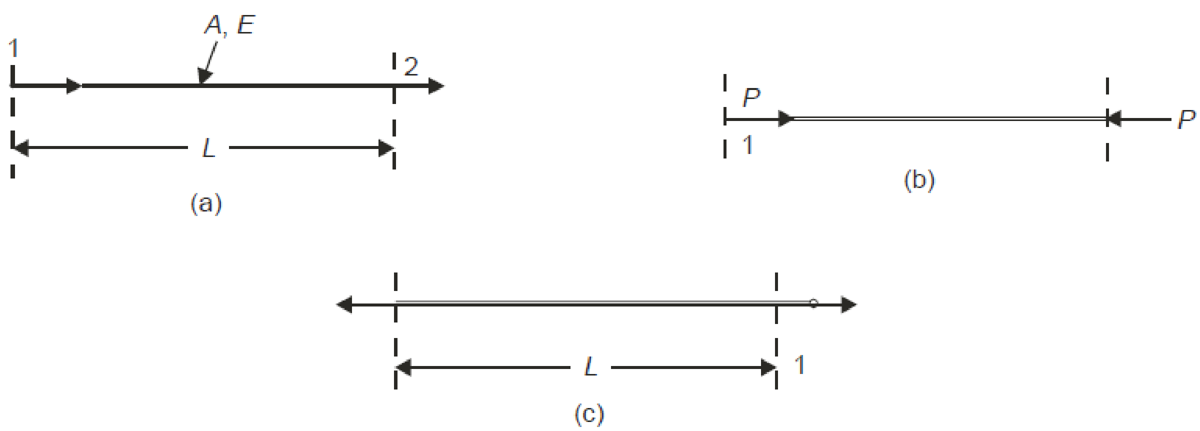


Figure 1.7. Stepped Bar subjected to axial force

For such bar with cross section A, Young's Modulus E and length L (Fig. 7 (a)) extension/shortening δ is given by

$$\delta = PL/AE$$

By giving unit displacement in coordinate direction 1, the forces development in the coordinate direction 1 and 2 can be found (Fig. 7 (b)). Hence from the definition of stiffness matrix,

$$k_{11} = \frac{EA}{L} and k_{21} = -\frac{EA}{L}$$

Similarly giving unit displacement in coordinate direction 2 (refer Fig. 7 (c)), we get,

$$k_{12} = -\frac{EA}{L} and k_{22} = \frac{EA}{L}$$

$$[k] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

5. Applications/ Simulation/ related Laboratory example

The matrix approach is applied stepped and tapered bars with structural load

6. MCQ- Post Test

1. The standard form of matrix displacement equation

   (a) $[k]\ \{u\} < \{F\}$

   (b) $[k]\ \{u\} > \{F\}$

   (c) $[k]\ \{u\}\ \{F\}=0$

   (d) $[k]\ \{u\} = \{F\}$

2. The extension of stepped bar mathematically represented as

   (a) δ = PL-AE
   (b) δ = PL+AE
   (c) δ = PL/AE
   (d) None of the above

6. Conclusions

- The matrix displacement method is successfully applied in stepped bar.

7. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

8. Video

https://www.youtube.com/watch?v=JFiBcVnAqMM

9. Assignments

1. Derive the stiffness matrix for stepped bar.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Application to the continuum**

1. Aim and Objectives:

- To understand the basics of continuum in three dimensional space

2. Pre-Test-MCQ type

1. Degrees of freedom means
   (a) Number of dependent coordinates required to describe a body
   (b) Number of independent coordinates required to describe a body
   (c) Force required to move a body in x-direction
   (d) None of the above
2. The point load acting
   (a) At a surface area
   (b) Along a line
   (c) At a point
   (d) None of the above

3. Prerequisites

The basics of engineering mechanics is required

4. Theory behind – Application to the continuum

A three –dimensional body occupying a volume V and and having a surface S is shown in Fig.1.8. points in the body are located by x,y,z co-ordinates. The boundary is constrained on some region, where displacement is specified, On part of the boundary, distributed force per unit area T, also called traction, is applied. Under the force, the body deforms. The deformation of a point x(=[x,y,z]^T given by the three components of its displacement.

$$u = [u,v,w]^T$$

The distributed force per unit volume, for example. the weight per unit volume, is the vector f given by

$$f = [f_x, f_y, f_z]^T$$

The body force acting on the elemental volume dV is shown in Fig1. 8. The surface traction T may be given by its component values at points on the surface:

$$T = [T_x, T_y, T_z]^T$$

Examples of traction are distributed contact force and action of pressure. A load P acting at a point *i* is represented by its three components:

$$P_i = [P_x, P_y, P_z]^T$$

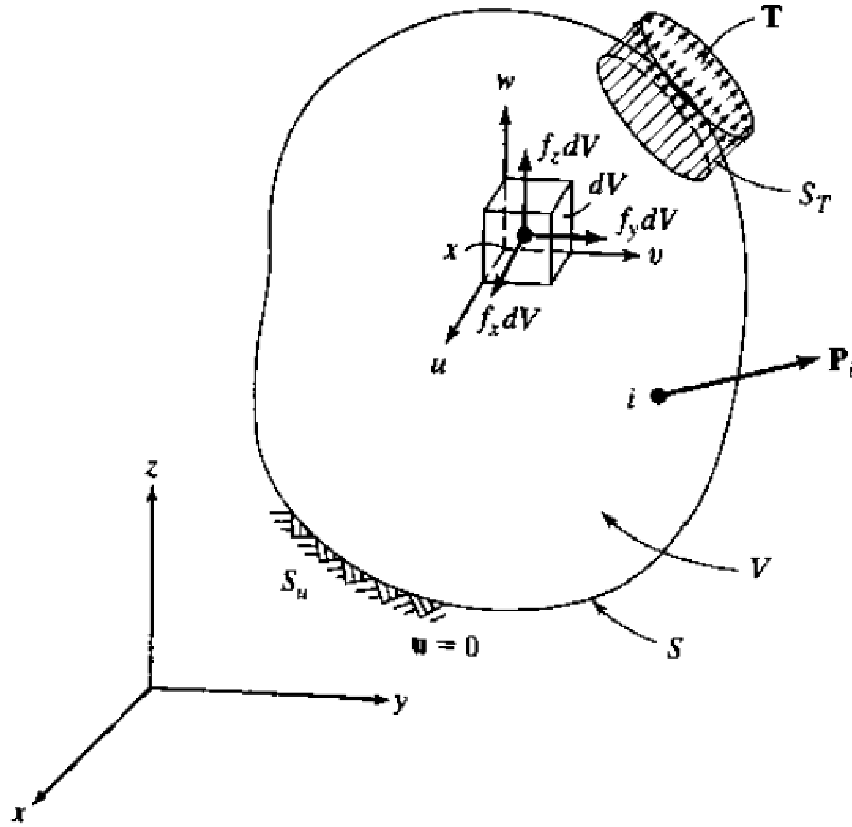


Figure 1.8. Three Dimensional body

5. Applications/ Simulation/ related Laboratory example

The application of continuum is applicable for all field of engineering.

6. Post Test- MCQ

1. The definition of Traction force is
   (a) distributed force per unit line
   (b) distributed force per unit area
   (c) distributed force per unit volume
   (d) none of the above
2. Example for traction force
   (a) Pressure
   (b) Temperature
   (c) Resistance
   (d) All of the above
3. The minimum number of dimensions are required to define the position of a point in space is:
   (a) one
   (b) two
   (c) three
   (d) four

7. Conclusions

In a three dimensional body, the various forces acting in a body is discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

9. Videos

https://www.youtube.com/watch?v=JDJtrUXzxck

10. Assignments

Write a short notes on application of continuum used in FEA approach.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Discretisation**

1. Aim and Objectives

To learn about the continuum Discretisation

2. Pre-Test-MCQ type

1. Initial conditions are used for __________ problems.
   (a) time-dependent problems
   (b) boundary value problems
   (c) control volume problems
   (d) finite difference problems
2. Which of these is the best practice regarding outlet boundaries?
   (a) Outlet boundaries should be at the exact outlet of the geometry
   (b) Outlet boundaries should be set as close as possible to the inlet boundaries
   (c) Outlet boundaries should be set as far as possible to the physical geometry
   (d) Outlet boundaries should be set as close as possible to the physical geometry

3. Pre-Requisites

The knowledge of various mechanical structures is required.

4. Theory behind – Discretisation

The process of modeling a structure using suitable number, shape and size of the elements is called discretization. The modeling should be good enough to get the results as close to actual behavior of the structure as possible.

Nodes at Discontinuities

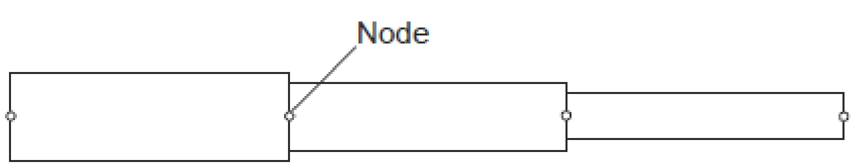
In a structure we come across the following types of discontinuities:

(a) Geometric
(b) Load
(c) Boundary conditions
(d) Material.

(a) Geometric Discontinuities

Wherever there is sudden change in shape and size of the structure there should be a node or line of nodes.

Figure 1.9(a&b). shows some of such situations.



(a)

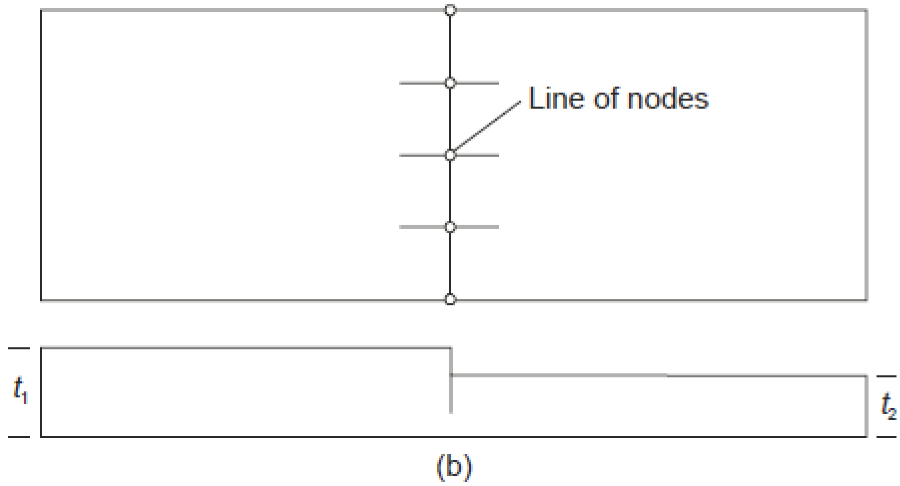


Figure 1.9. (a) Bar subject to axial forces (b) Plate with varying

(b) Discontinuity of Loads

Concentrated loads and sudden change in the intensity of uniformly distributed loads are the sources of discontinuity of loads. A node or a line of nodes should be there to model the structure. Some of these situations are shown in Fig1.10.

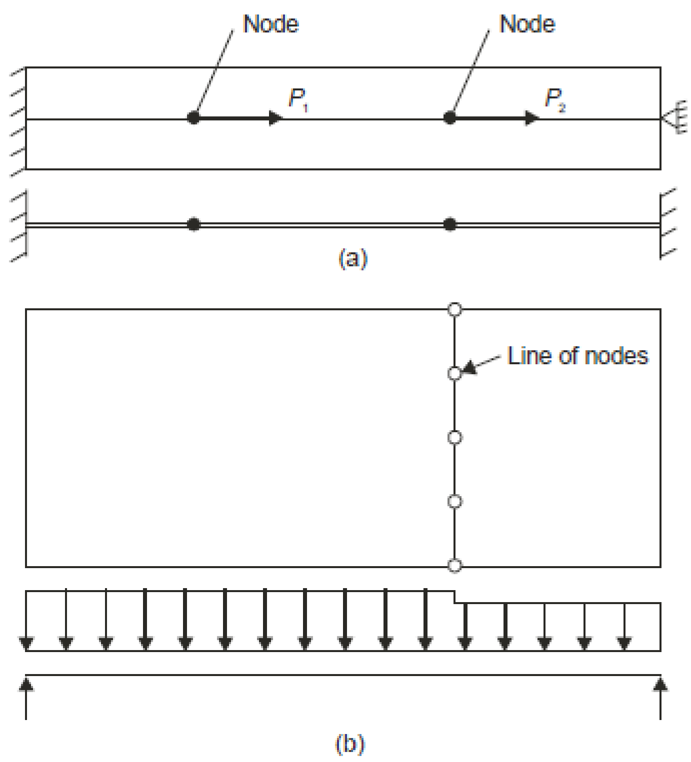


Figure.1.10 (a) FEM model (b) Slab with different UDLs

(c)Discontinuity of Boundary conditions

If the boundary condition for a structure suddenly change we have to discretize such that there is node or a line of nodes. This type of situations are shown in Fig. 11

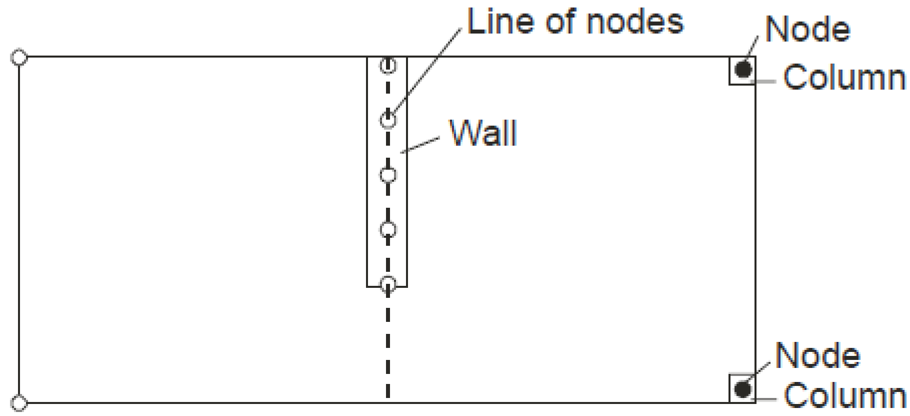


Figure1.11 Slab with intermediate wall and columns

(d) Material Discontinuity
Node or node lines should appear at the places where material discontinuity is seen.

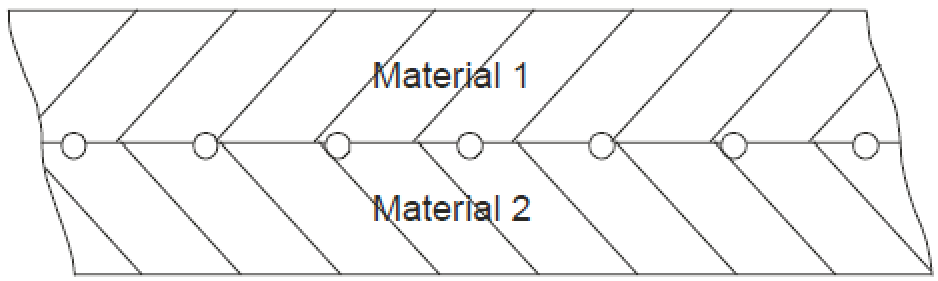


Figure1.12 Material Discontinuity

Example for irregular domain

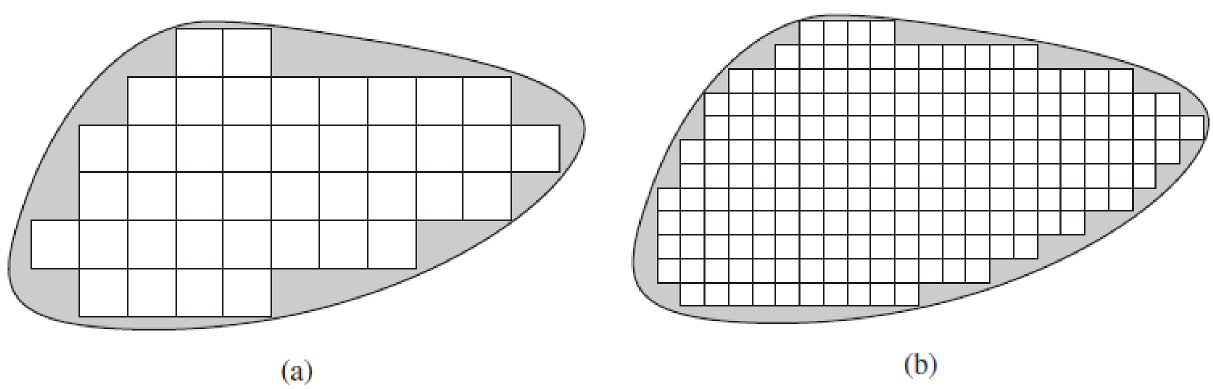


Figure1.13. (a) Arbitrary curved-boundary domain modeled using square elements. Stippled areas are not included in the model. A total of 41 elements is shown. (b) Refined
finite element mesh showing reduction of the area not included in the model. A total of 192 elements is shown.

The process of representing a physical domain with finite elements is referred to as Discretisation, and the resulting set of elements is known as the finite element mesh. As most of the commonly used element geometries have straight sides, it is generally impossible to include the entire physical domain in the element mesh if the domain includes curved boundaries. Such a situation is shown in Figure 1.13a, where a curved-boundary domain is meshed (quite coarsely) using square elements. A refined mesh for the same domain is shown in Figure 1.13b, using smaller, more numerous elements of the same type. Note that the refined mesh includes significantly more of the physical domain in the finite element representation and the curved boundaries are more closely approximated.

5. Applications/ Simulation/ related Laboratory example

The discretisation is mainly used as a beginning procedure for FEA problems.

6. MCQ-Post test

1. The art of subdividing a structure into convenient number of smaller components is known as
   (a) global stiffness matrix
   (b) force vector
   (c) discretization
   (d) none
2. All the calculations are made at limited number of points known as
   (a) Elements
   (b) Nodes
   (c) Discretization
   (d) Mesh

3. Domain is divided into some segment is called
   (a) Element
   (b) Node
   (c) Segment
   (d) Points

4. Finite element is -------------------
   (a) Small unit having definite shape and nodes
   (b) Small unit having definite shape and no nodes
   (c) Small unit only
   (d) Only nodes

7. Conclusion

The discretization of different mechanical structure were discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=ambbGRqMeJU

10. Assignments

Explain the Nodes at Discontinuities With suitable number of examples

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Matrix algebra**

1. Aim and Objectives

To familiarize about the Matrix algebra

2. Pre-Test-MCQ type

1. The determinant of identity matrix is?
   (a) 1
   (b) 0
   (c) Depends on the matrix
   (d) None of the mentioned
2. Which of the following property of matrix multiplication is correct?
   (a) Multiplication is not commutative in general
   (b) Multiplication is associative
   (c) Multiplication is distributive over addition
   (d) All of the mentioned

3. If A is a lower triangular matrix then A^T is a __________
   (a) Lower triangular matrix
   (b) Upper triangular matrix
   (c) Null matrix
   (d) None of the mentioned

3. Pre-Requisites

The knowledge of basics of matrix operations is required.

4. Theory behind – Matrix algebra

The study of matrices here is largely motivated from the need to solve systems of simultaneous equations of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

where $x_1, x_2, \ldots x_n$ are the unknowns. The above can be conveniently expressed in matrix form as

Ax=b

where A is a square matrix of dimensions $(n \times n)$, and x and b are vectors of dimension $(n \times 1)$, given as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

From this information, we see that a matrix is simply an array of elements. The matrix is simply an array of elements. A is also denoted as [A]. An element located at the ith row and jth column of A is denoted by a_{ij}.

The analysis of engineering problems by the finite element method involves a sequence of matrix operations, This fact allows us to solve large-scale problems because computers are ideally suited for matrix operations.

Row and Column Vectors

A matrix of dimension ($1 \times n$) is called a row vector, while a matrix of dimension ($m \times 1$) is called a column vector. For example,

D=[1 -1 2]

is a (1×3) row vector, and

$$e = \begin{Bmatrix} 2 \\ 2 \\ -6 \\ 0 \end{Bmatrix} \text{ is a (4×1) column vector.}$$

Addition and Subtraction

Consider two matrices A and B, both of dimension ($m \times n$). Then, the sum C=A+B is defined as

$$c_{ij} = a_{ij} + b_{ij}$$

That is, the (ij)th component of C is obtained by adding the (ij)th component of A to the (ij)th component of B. For example,

$$\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$$

Subtraction is similarly defined.

Matrix Multiplication

The product of an (m ×n) matrix A and an (n × p) matrix Bresults in an (m × p) matrix C. That is,

$$\underset{(m\times n)}{A} \quad \underset{(n\times p)}{B} = \underset{(m\times p)}{C}$$

The (ij)th component of C is obtained by taking the dot product

$$c_{ij} = (i\text{th row of A}) \cdot (j\text{th column of B})$$

For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 5 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ -10 & 7 \end{bmatrix}$$

(2×3) (3×2) (2×2)

Square Matrix

A matrix whose number of rows equals the number of columns is called a square matrix.

Diagonal Matrix

A diagonal matrix is a square matrix with nonzero elements only along the principal diagonal. For example,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

5. Applications/ Simulation/ related Laboratory example

In general, the applications of matrix is widely applied in many fields.

6. Post Test- MCQ

1. The transpose of a column matrix is
   (a) zero matrix
   (b) diagonal metrix
   (c) column matrix
   (d) row matrix
2. Two matrics Ana dB are multiplied to get AB if
   (a) both are rectangular
   (b) both have same order
   (c) no of columns of A is equal to columns of B
   (d) no of rows of A is equal to no of columns of B
3. Let A = [0 1 0 0], A^{-1} is equal to __________
   (a) Null matrix
   (b) Identity matrix
   (c) Does not exist
   (d) None of the mentioned

7. Conclusion

The knowledge of matrix algebra is effectively familiarized.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India

9. Video

https://www.youtube.com/watch?v=tVckr5GiUek

10.Assignments

1. Write the basic matrix operations with suitable examples.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Gaussian elimination**

1. Aim and Objectives

To understand the a method of Gaussian elimination

2. Pre-Test-MCQ type

1. What is the order of a matrix?
   - **(a) number of rows X number of columns**
   - (b) number of columns X number of rows
   - (c) number of rows X number of rows
   - (d) number of columns X number of columns
2. Matrix A when multiplied with Matrix C gives the Identity matrix I, what is C?
   - (a) Identity matrix
   - **(b) Inverse of A**
   - (c) Square of A
   - (d) Transpose of A

3. Prerequisites

The basics of matrix operation must be known

4. Theory behind – Gaussian elimination

Consider a linear system of simultaneous equations in matrix form as

Ax=b

where $\mathbf{A}$ is $(n \times n)$ and $\mathbf{b}$ and $\mathbf{x}$ are $(n \times 1)$. If det $\mathbf{A} \neq 0$, then we can premultiply both sides of the equation by $\mathbf{A}^{-1}$ to write the unique solution for $\mathbf{x}$ as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. However, the explicit construction of $\mathbf{A}^{-1}$, say, by the cofactor approach, is computationally expensive and prone to round-off errors. Instead, an elimination scheme is better. The powerful Gaussian elimination approach for solving $\mathbf{Ax} = \mathbf{b}$ is discussed in the following pages.

Gaussian elimination is the name given to a well-known method of solving simultaneous equations by successively eliminating unknowns. We will first present the methodby means of an example, followed by a general solution and algorithm. Consider the simultaneous equations

$$\begin{aligned} x_1 - 2x_2 + 6x_3 &= 0 \quad (\text{I}) \\ 2x_1 + 2x_2 + 3x_3 &= 3 \quad (\text{II}) \\ -x_1 + 3x_2 \qquad &= 2 \quad (\text{III}) \end{aligned} \tag{2.23}$$

The equations are labeled as I, II, and III. Now, we wish to eliminate x_1 from II and III. We have, from Eq. I, $x_1 = +2x_2 - 6x_3$. Substituting for x_1 into Eqs. II and III yields

$$
\begin{aligned}
x_1 - 2x_2 + 6x_3 &= 0 \quad (\mathrm{I}) \\
0 + 6x_2 - 9x_3 &= 3 \quad (\mathrm{II}^{(1)}) \\
0 + x_2 + 6x_3 &= 2 \quad (\mathrm{III}^{(1)})
\end{aligned} \tag{2.24}
$$

It is important to realize that Eq. 2.24 can also be obtained from Eq. 2.23 by **row operations.** Specifically, in Eq. 2.23, to eliminate x_1 from II, we subtract 2 times I from II, and to eliminate x_1 from III we subtract −1 times I from III. The result is Eq. 2.24. Notice the zeroes below the main diagonal in column 1, representing the fact that x_1 has been eliminated from Eqs. II and III. The superscript (1) on the labels in Eqs. 2.24 denotes the fact that the equations have been modified once.

We now proceed to eliminate x_2 from III in Eqs. 2.24. For this, we subtract $\frac{1}{6}$ times II from III. The resulting system is

$$
\begin{bmatrix}
x_1 - 2x_2 + 6x_3 = 0 \\
0 + 6x_2 - 9x_3 = 3 \\
0 \quad 0 \quad \frac{15}{2}x_3 = \frac{3}{2}
\end{bmatrix}
\begin{matrix} (\mathrm{I}) \\ (\mathrm{II}^{(1)}) \\ (\mathrm{III}^{(2)}) \end{matrix} \tag{2.25}
$$

The coefficient matrix on the left side of Eqs. 2.25 is upper triangular. The solution now is virtually complete, since the last equation yields $x_3 = \frac{1}{5}$, which, upon substitution into the second equation, yields $x_2 = \frac{4}{5}$, and then $x_1 = \frac{2}{5}$ from the first equation. This process of obtaining the unknowns in reverse order is called **back-substitution.**

These operations can be expressed more concisely in matrix form as follows: Working with the augmented matrix $[\mathbf{A}, \mathbf{b}]$, the Gaussian elimination process is

$$
\begin{bmatrix} 1 & -2 & 6 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & 3 & 0 & 2 \end{bmatrix} \rightarrow
\begin{bmatrix} 1 & -2 & 6 & 0 \\ 0 & 6 & -9 & 3 \\ 0 & 1 & 6 & 2 \end{bmatrix} \rightarrow
\begin{bmatrix} 1 & -2 & 6 & 0 \\ 0 & 6 & -9 & 3 \\ 0 & 0 & 15/2 & 3/2 \end{bmatrix} \tag{2.26}
$$

which, upon back-substitution, yields

$$
x_3 = \tfrac{1}{5} \qquad x_2 = \tfrac{4}{5} \qquad x_1 = \tfrac{2}{5} \tag{2.27}
$$

5. Applications/ Simulation/ related Laboratory example

The main application is element model description in all FEA problems.

6. MCQ-Post test

1. Find the values of x, y, z in the following system of equations by gauss elimination method.
   (a) 2x + y – 3z = -10
   (b) -2y + z = -2
   (c) z = 6
   (d) x+y-z=9

2. In Gaussian elimination method, original equations are transformed by using

(a) Column operations
(b) Row Operations
(c) Mathematical Operations
(d) Subset Operation

7. Conclusion

The element model based gauss elimination is interpreted.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=NoqBkHNzDrE

10. Assignments

1. Solve this system of equations and comment on the nature of the solution using Gauss Elimination method.

$$x + y + z = 0$$
$$-x - y + 3z = 3$$
$$-x - y - z = 2$$

2. Solve the below equation using Gauss-Elimination method.

$$3x + y - z = 3$$
$$2x - 8y + z = -5$$
$$x - 2y + 9z = 8$$

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Governing equations for continuum**

1. Aim and Objectives

To familiarize on the Governing equations for continuum

2. Pre-Test-MCQ type

1. Resilience can also be termed as ___________

(a) Stress energy
(b) Strain energy
(c) Modulus
(d) Tenacity

2. When a body falls freely towards the earth, then its total energy

(a) Decreases
(b) Increases
(c) First increases and then decreases
(d) Remains constant

3. Prerequisites

To know about the fundamentals of engineering mechanics

4. Theory behind – Governing equations for continuum

Potential Energy, Π

The total potential energy Π of an elastic body, is defined as the sum of total strain energy (U) and the work potential:

$$\Pi = \underset{(U)}{\text{Strain energy}} + \underset{(WP)}{\text{Work potential}} \tag{1.24}$$

For linear elastic materials, the strain energy per unit volume in the body is $\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon}$. For the elastic body shown in Fig. 1.1, the total strain energy U is given by

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T\boldsymbol{\epsilon}\, dV \tag{1.25}$$

The work potential WP is given by

$$WP = -\int_V \mathbf{u}^T \mathbf{f}\, dV - \int_S \mathbf{u}^T \mathbf{T}\, dS - \sum_i \mathbf{u}_i^T \mathbf{P}_i \qquad (1.26)$$

The total potential for the general elastic body shown in Fig. 1.1 is

$$\Pi = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\epsilon}\, dV - \int_V \mathbf{u}^T \mathbf{f}\, dV - \int_S \mathbf{u}^T \mathbf{T}\, dS - \sum_i \mathbf{u}_i^T \mathbf{P}_i \qquad (1.27)$$

We consider conservative systems here, where the work potential is independent of the path taken. In other words, if the system is displaced from a given configuration and brought back to this state, the forces do zero work regardless of the path. The potential energy principle is now stated as follows:

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

5. Applications/ Simulation/ related Laboratory example

The Governing equations for continuum is mainly used in all FEA problems

6. MCQ-Post test

1. Total potential energy is equal to
   (a) strain energy -work potential
   (b) strain energy /work potential
   (c) strain energy ×work potential
   (d) strain energy +work potential

2. The spring will have maximum potential energy when
   (a) it is pulled out
   (b) it is compressed
   (c) both (a) and (b)
   (d) neither (a) nor (b)

7. Conclusion

The Governing equations for continuum based on principle of potential energy is discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=xZUYjnld5rU

10. Assignments

1. Write short notes on potential energy application with some examples.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Classical Techniques in FEM (weighted residual method)**

1. Aim and Objectives

To understand the weighted residual method for numerical approximation

2. Pre-Test-MCQ type

1. The differential equation $2\frac{dy}{dx} + x^2 y = 2x + 3,\ y(0) = 5$ is
   - **(a) linear**
   - (b) nonlinear
   - (c) linear with fixed constants
   - (d) undeterminable to be linear or nonlinear

2. A differential equation is considered to be ordinary if it has
   - (a) one dependent variable
   - (b) more than one dependent variable
   - **(c) one independent variable**
   - (d) more than one independent variable

3. Prerequisites

The basic knowledge of engineering mathematics is required.

4. Theory behind –

Classical Techniques in FEM (weighted residual method)

It is a basic fact that most practical problems in engineering are governed by differential equations. Owing to complexities of geometry and loading, rarely are exact solutions to the governing equations possible. Therefore, approximate techniques for solving differential equations are indispensable in engineering analysis. Indeed, the finite element method is such a technique. However, the finite element method is based on several other, more-fundamental, approximate techniques, one of which is discussed in detail in this section and subsequently applied to finite element formulation.

The method of weighted residuals (MWR) is an approximate technique for solving boundary value problems that utilizes trial functions satisfying the prescribed boundary conditions and an integral formulation to minimize error, in an average sense, over the problem domain. The general concept is described here in terms of the one-dimensional case but, as is shown in later chapters, extension to two and three dimensions is relatively straightforward. Given a differential equation of the general form

$$D[y(x), x] = 0 \qquad a < x < b \tag{5.1}$$

subject to homogeneous boundary conditions

$$y(a) = y(b) = 0 \tag{5.2}$$

the method of weighted residuals seeks an approximate solution in the form

$$y^*(x) = \sum_{i=1}^{n} c_i N_i(x) \tag{5.3}$$

where y^* is the approximate solution expressed as the product of c_i unknown, constant parameters to be determined and $N_i(x)$ trial functions. The major requirement placed on the trial functions is that they be *admissible functions;* that is, the trial functions are continuous over the domain of interest and satisfy the specified boundary conditions exactly. In addition, the trial functions should be selected to satisfy the "physics" of the problem in a general sense. Given these somewhat lax conditions, it is highly unlikely that the solution represented by Equation 5.3 is exact. Instead, on substitution of the assumed solution into the differential Equation 5.1, a residual error (hereafter simply called *residual*) results such that

$$R(x) = D[y^*(x), x] \neq 0 \tag{5.4}$$

where $R(x)$ is the residual. Note that the residual is also a function of the unknown parameters c_i. The method of weighted residuals requires that the unknown parameters c_i be evaluated such that

$$\int_a^b w_i(x) R(x)\, \mathrm{d}x = 0 \qquad i = 1, n \tag{5.5}$$

where $w_i(x)$ represents n arbitrary weighting functions. We observe that, on integration, Equation 5.5 results in n algebraic equations, which can be solved for the n values of c_i. Equation 5.5 expresses that the sum (integral) of the weighted residual error over the domain of the problem is zero. Owing to the requirements placed on the trial functions, the solution is exact at the end points (the boundary conditions must be satisfied) but, in general, at any interior point the residual error is nonzero. As is subsequently discussed, the MWR may capture the exact solution under certain conditions, but this occurrence is the exception rather than the rule.

Several variations of MWR exist and the techniques vary primarily in how the weighting factors are determined or selected. The most common techniques are point collocation, subdomain collocation, least squares, and Galerkin's method [1]. As it is quite simple to use and readily adaptable to the finite element method, we discuss only Galerkin's method.

In Galerkin's weighted residual method, the weighting functions are chosen to be identical to the trial functions; that is,

$$w_i(x) = N_i(x) \qquad i = 1, n \tag{5.6}$$

Therefore, the unknown parameters are determined via

$$\int_a^b w_i(x)R(x)\,\mathrm{d}x = \int_a^b N_i(x)R(x) = 0 \qquad i = 1, n \tag{5.7}$$

again resulting in n algebraic equations for evaluation of the unknown parameters. The following examples illustrate details of the procedure.

Example on Galerkin's Problem

Use Galerkin's method of weighted residuals to obtain a one-term approximation to the solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 4x \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = 0$, $y(1) = 1$.

■ Solution

Here the boundary conditions are not homogeneous, so a modification is required. Unlike the case of homogeneous boundary conditions, it is not possible to construct a trial solution of the form $c_1 N_1(x)$ that satisfies both stated boundary conditions. Instead, we assume a trial solution as

$$y^* = c_1 N_1(x) + f(x)$$

where $N_1(x)$ satisfies the homogeneous boundary conditions and $f(x)$ is chosen to satisfy the nonhomogeneous condition. (Note that, if both boundary conditions were nonhomogeneous, two such functions would be included.) One such solution is

$$y^* = c_1 x(x - 1) + x$$

which satisfies $y(0) = 0$ and $y(1) = 1$ identically.

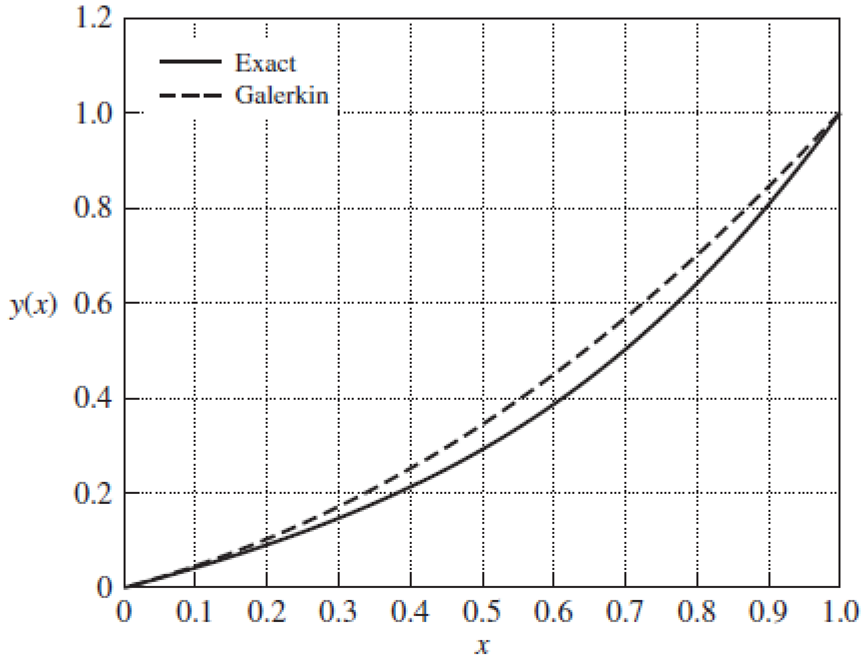


Figure 5.3 Solutions to Example 5.3.

Substitution into the differential equation results in the residual

$$R(x; c_1) = \frac{d^2 y^*}{dx^2} + y^* - 4x = 2c_1 + c_1 x^2 - c_1 x + x - 4x = c_1 x^2 - c_1 x + 2c_1 - 3x$$

and the weighted residual integral becomes

$$\int_0^1 N_1(x) R(x; c_1)\, dx = \int_0^1 x(x-1)(c_1 x^2 + c_1 x - 2c_1 - 3x)\, dx = 0$$

While algebraically tedious, the integration is straightforward and yields

$$c_1 = 5/6$$

so the approximate solution is

$$y^*(x) = \frac{5}{6} x(x-1) + x = \frac{5}{6} x^2 + \frac{1}{6} x$$

As in the previous example, we have the luxury of comparing the approximate solution to the exact solution, which is

$$y(x) = 4x - 3.565 \sin x$$

The approximate solution and the exact solution are shown in Figure 5.3 for comparison. Again, the agreement is observed to be reasonable but could be improved by adding a second trial function.

5. Applications/ Simulation/ related Laboratory example

The galerkin's method is applicable to solve all non-structural problems(Exmple: fluid and heat transfer applications)

6. MCQ-Post test

1. For Non-structural problems, which method is commonly preferred
   (a) Rayleigh-Ritz Method
   (b) Galerkin Method
   (c) Runge kutta Method
   (d) None of these

2. Which function mainly considered in Galerkin approach

   (a) Polynomial Function
   (b) Trial function
   (c) Polynomial and Trial function
   (d) None of these

3. The Trial function in Galerkin approach contains a_1, a_2 and so on. Therefore the name of a_1, a_2…an refers
   (a) Galerkin parameter
   (b) Ritz parameter
   (c) Both Galerkin & Ritz parameters
   (d) None of these

4. For solving of fluid mechanics problems, the essential boundary conditions are
   (a) Compulsory
   (b) Not Compulsory
   (c) Partially Compulsory
   (d) None of these

5. The examples for non-structural problems
   (a) Heat flow
   (b) Fluid flow
   (c) Both heat and fluid flow
   (d) None of these

7. Conclusion

The Galerkin's method and related example are effectively discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=SAqCB3JIw0M

10.Assignments

1. The following differential equation is available for a physical phenomenon:

$$\frac{d^2y}{dx^2} - 10x^2 = 5; 0 \le x \le 1$$

The boundary conditions are: y(0)=0 and y(1)=1. By using Galerkin's method of weighted residuals to find an approximate solution of the above differential equation and also compare with exact solutions.

2. The differential equation of a physical phenomenon is given by,

$$\frac{d^2y}{dx^2} + 500x^2 = 5, 0 \le x \le 1.$$

Use the Trial function, $y = a_1(x - x^4)$. The boundary conditions are: y(0)=0 and y(1)=0. Calculate the value of the parameter a_1 by the Galerkin's approach.

UNIT-1

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Introduction**

Name of the Topic : **Rayleigh-Ritz method**

1. Aim and Objectives:

- To understand the theory of elasticity including strain and displacement
- To analyze solid mechanics problems using classical methods and energy methods

2. Outcomes

At the end of the topic, a student will be able to

- apply the Rayleigh-Ritz method to solve structural problems and outline the requirements for convergence.

3. Pre-Requisites

- The students should have a basic knowledge of mathematics, and mechanics of solids.
- It is assumed that the student has knowledge about basic calculus and differential equations.

1. 1-$\cos^2$A is equal to:

   (a)$\sin^2$A

   (b)$\tan^2$A

   (c)1-$\sin^2$A

   (d)$\sec^2$A

2. A partial differential equation has

   (a) one independent variable
   (b) two or more independent variables
   (c) more than one dependent variable
   (d) equal number of dependent and independent variables

3. Strain energy is the

   (a) energy stored in a body when strained within elastic limits
   (b) energy stored in a body when strained upto the breaking of a specimen
   (c) maximum strain energy which can be stored in a body
   (d) proof resilience per unit volume of a material

4. A beam is loaded as cantilever. If the load at the end is increased, the failure will occur

(a) In the middle

(b) At the tip below the load

(c) At the support

(d) Anywhere

5. A simply supported beam of span '*l*' meters carries a UDL of '*w*' per unit length over the entire span, the maximum bending moment occurs at ______

(a) At point of contra flexure

(b) Centre

(c) End supports

(d) Anywhere on the beam

6. ________ is a horizontal structural member subjected to transverse loads perpendicular to its axis.

(a) Strut

(b) Column

(c) Beam

(d) Truss

7. Units of U.D.L?

(a) KN/m

(b) KN-m

(c) KN-m×m

(d) KN

8. In simply supported beam deflection is maximum at _____________

(a) Midspan

(b) Supports

(c) Point of loading

(d) Through out

9. Which of the following is a differential equation for deflection?

(a) $dy/dx=(M/EI)$

(b) $dy/dx=(MI/E)$

(c)$d^2y/dx^2 =(M/EI)$

(d) $d^2y / dx^2 = (ME/I)$

10. Macaulay's method is used to determine ______

(a) deflection

(b) strength

(c) toughness

(d) all of the above

4. Theory behind – Rayleigh-Ritz Method

The Rayleigh–Ritz method of expressing field variables by approximate method clubbed with minimization of potential energy has made a big breakthrough in finite element analysis. In 1870 Rayleigh used an approximating field with single degree of freedom for studies on vibration problems. In 1909 he used approximating field with several functions, each function satisfying boundary conditions and associating with separate degree of freedom. Ritz applied this technique to static equilibrium and Eigen value problems.

The procedure for static equilibrium problem is given below:

Consider an elastic solid subject to a set of loads. The displacements and stresses are to be determined. Let *u, v* and *w* be the displacements in *x, y* and *z* coordinate directions. Then for each of displacement component an approximate solution is taken as

$$u = \sum a_i \phi_i (x, y, z) \text{ for } i = 1 \text{ to } m_1$$

$$v = \sum a_j \phi_j (x, y, z) \text{ for } j = m_1 + 1 \text{ to } m_2 \qquad \ldots(9.12)$$

$$w = \sum a_k \phi_k (x, y, z) \text{ for } k = m_2 + 1 \text{ to } m$$

The function φ_i are usually taken as polynomials satisfying the boundary conditions. '*a*' are the amplitudes of the functions. Thus in equation 9.12 there are n number of unknown '*a*' values. Substituting these expressions for displacement in strain displacements and stress strain relations, potential energy expression 9.16 can be assembled. Then the total potential energy

$$\Pi = \Pi \left(a_1, a_2 \ldots a_{m1}, a_{m1+1} \ldots a_{m2}\, a_{m2+1} \ldots a_m\right)$$

From the principle of minimum potential energy,

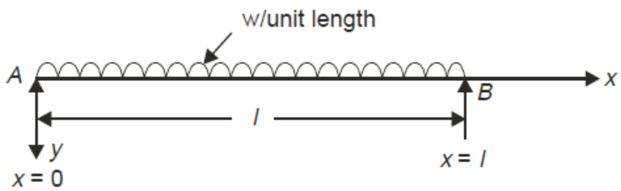
$$\frac{d\Pi}{da_i} = 0 \text{ for } i = 1 \text{ to } m. \qquad \ldots(9.22)$$

From the solution of *m* equation of 9.22, we get the values of all '*a*' . With these values of 'a_i's and φ_i 's satisfying boundary conditions, the displacements are obtained. Then the strains and stresses can be assembled.

The Rayleigh – Ritz procedure is illustrated with structural problems below:

Example 9.5: Using Ragleigh–Ritz method determine the expressions for deflection and bending moments in a simply supported beam subjected to uniformly distributed load over entire span. Find the deflection and moment at midspan and compare with exact solutions.

Solution: Figure 9.8 shows the typical beam. The Fourier series $y = \sum_{m=1,3}^{\alpha} a_i \sin\frac{m\pi x}{l}$ is the ideal function for simply supported beams since $y = 0$ and $M = EI\frac{d^2y}{\partial x^2} = 0$ at $x = 0$ and $x = l$ are satisfied. For the simplicity



let us consider only two terms in the series i.e. let

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$\Pi = \int_0^l \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx - \int_0^l wy\, dx$$

Substituting y in equation (b) we get

$$= \int_0^l \frac{EI}{2}\left[-\frac{\pi^2}{l^2}a_1 \sin\frac{\pi x}{l} - \frac{9\pi^2}{l^2}a_2 \sin\frac{3\pi x}{l}\right]^2 dx - \int_0^l w\left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right) dx$$

$$= \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l\left(a_1 \sin\frac{\pi x}{l} + 9a_2 \sin\frac{3\pi x}{l}\right)^2 dx - w\left[-a_1\frac{l}{\pi}\cos\frac{\pi x}{l} - a_2\frac{l}{3\pi}\cos\frac{3\pi x}{l}\right]_0^l$$

$$= \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l\left(a_1^2 \sin^2\frac{\pi x}{l} + 18a_1a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l} + 81a_2^2 \sin^2\frac{3\pi x}{l}\right) dx - \frac{wl}{\pi}\left[2a_1 + \frac{2a_2}{3\pi}\right]$$

Noting that $\int_0^l \sin^2 \frac{\pi x}{l} dx = \int_0^l \frac{1}{2}\left(1 - \cos\frac{2\pi x}{l}\right) dx = \frac{1}{2}$

$$\int_0^l \sin\frac{\pi x}{l} \sin\frac{3\pi x}{l} dx = \int_0^l \left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}\right) dx = 0$$

and $$\int_0^l \sin^2 \frac{3\pi x}{l} = \int_0^l \frac{1}{2}\left(1 - \cos\frac{6\pi x}{l}\right) dx = \frac{1}{2}$$

we get, $$y = \frac{EI}{2}\frac{\pi^4}{l^4}\left(a_1^2 \frac{l}{2} + 81\frac{l}{2} a_2^2\right) - \frac{2wl}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

$$= \frac{EI}{4}\frac{\pi^4}{l^3}\left(a_1^2 + 81a_2^2\right) - \frac{2wl}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

Π to be minimum,

$$\frac{\partial \Pi}{\partial a_1} = 0 \text{ and } \frac{\partial \Pi}{\partial a_2} = 0.$$

i.e., $$\frac{EI\pi^4}{4l^3} 2a_1 - \frac{2wl}{\pi} = 0$$

or $$a_1 = \frac{4wl^4}{EI\pi^5}$$

and $$\frac{EI\pi^4}{4l^3} 81 \times 2a_2 - \frac{2wl}{3\pi} = 0$$

or $$a_2 = \frac{4wl^4}{243EI\pi^5}$$

$$\therefore\ y = \frac{4wl^4}{EI\pi^5}\sin\frac{\pi x}{l} + \frac{4wl^4}{243EI\pi^5}\sin\frac{3\pi x}{l}$$

$\therefore$ Max. deflection which occurs at $x = \frac{l}{2}$ is

$$y_{\max} = \frac{4wl^4}{EI\pi^5} - \frac{4wl^4}{243EI\pi^5} = \frac{wl^4}{76.82EI}$$

Thus the deflection is almost exact.

Now, $$M_x = EI\frac{d^2y}{dx^2} = EI\left(-a_1\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} - a_2\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right)$$

$$= EI\left[-\frac{4wl^2}{EI\pi^3}\sin\frac{\pi x}{l} - \frac{4wl^2 x9}{243EI\pi^3}\sin\frac{3\pi x}{l}\right]$$

$$M_{centre} = \left[-\frac{4wl^2}{EI\pi^3} + \frac{4wl^2 x9}{243EI\pi^3}\right] = \frac{wl^2}{8.05}$$

we know the exact value is $\frac{wl^2}{8}$.

By taking more terms in Furier series more accurate results can be obtained.

5. MCQ- Post Test

1. Total potential energy is equal to
   - (a) strain energy -work potential
   - (b) strain energy /work potential
   - (c) strain energy ×work potential
   - **(d) strain energy +work potential**
2. Rayleigh Ritz Method is applicable for
   - **(a) Structural problems**
   - (b) Fluid mechanics problems
   - (c) Both Structural and Fluid mechanics problems
   - (d) None of These
3. Convergence is a process of
   - a. Dividing the domain
   - b. Converting local coordinates into natural coordinates
   - **c. Arriving at a solution that is close to the exact solution**
   - d. Arriving at a solution that is far from the exact solution

4. A cantilever beam subjected to uniformly distributed load problems solved by
   - (a) Galerkin Method
   - **(b) Rayleigh-Ritz Method**
   - (c) Both Galerkin and Rayleigh-Ritz Method
   - (d) None of these

5. In Rayleigh-Ritz Method, which series is considered for approximating function
   (a) Laplace series
   (b) Inverse Fourier series
   (c) Inverse laplace series
   (d) Fourier series

6. Finding of Ritz parameter for structural problems is
   (a) Essential
   (b) Not essential
   (c) Partially essential
   (d) Partially not essential

7. Conclusions

- The Rayleigh–Ritz method is a direct method to find an approximate solution for boundary value problems.
- Useful for solving Structural mechanics problems
- It is also known as variational approach

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.

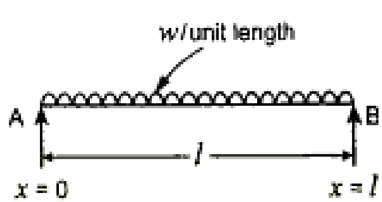
- BHAVIKATTI S.S."Finite Element Analysis", New Age International Publishers, 2005, India

9. Video

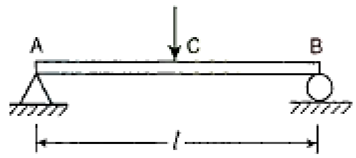
https://www.youtube.com/watch?v=-g8mb9ihXH0

10. Assignments

- A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at mids-span by using Rayleigh- Ritz method and compare with exact solutions.



- A beam AB of span '*l*' simply supported at ends and carrying a concentrated load W at the centre '*C*' as shown in Figure. Determine the deflection at midspan by using Rayleigh- Ritz method and compare with the exact solutions.


A
C
B
l

UNIT-2

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **One Dimensional Problems**

Name of the Topic : **Finite element modeling**

1. Aim and Objectives

- To understand the use of FEM to a range of Engineering Problems
- To apply one dimensional finite element method to solve bar and truss type problems

2. Pre-Test-MCQ type

1. The Force required to produce unit displacement is
   (a) Pressure
   (b) Traction
   (c) Stiffness
   (d) None

2. The materials having same elastic properties in all directions arc called
   (a) Ideal materials
   (b) Uniform materials
   (c) Isotropic materials
   (d) Paractical materials

3. The ultimate tensile stress of mild steel compared to ultimate compressive stress is
   (a) Same
   (b) More
   (c) Less
   (d) More or less depending on other factors

4. Prerequisites

The knowledge of strength of materials is required.

5. Theory behind – Finite element modeling

The total potential energy and the stress-strain and strain-displacement relationships are now used in developing the finite element method for a one-dimensional problem. The basic procedure is the same for two- and three-dimensional problems discussed later in the book. For the one-dimensional problem, the stress, strain, displacement, and loading depend only on the variable x. That is, the vectors u,σ,ε, T, and f expressed as

$u=u(x) \quad \sigma=\sigma(x) \quad \varepsilon=u(x) \quad T=T(x)$ and $f=f(x)$

Furthermore, the stress-strain and strain-displacement relations are

$\sigma = E\varepsilon$ and $\varepsilon = du/dx$

For one-dimensional problems, the differential volume dV can be written as

$dV = A\,dx$

The loading consists of three types: the body force *f,* the traction forte *T,* and the point load p_i;. These forces are shown acting on a body in Fig. 2.1. A body force is a distributed force acting on every elemental volume of the body and has the units of force per unit volume. The self-weight due to gravity is an example of a body force. A traction force is a distributed load acting on the surface of the body. For the one-dimensional problem considered here, however, the traction force is defined as force per unit length. This is done by taking the traction force to be the product of the force per unit area with the perimeter of the cross section. Frictional resistance, viscous drag, and surface shear are examples of traction forces in one-dimensional problems. Finally, P_i; is a force acting at a point *i* and *u,* is the *x* displacement at that point.

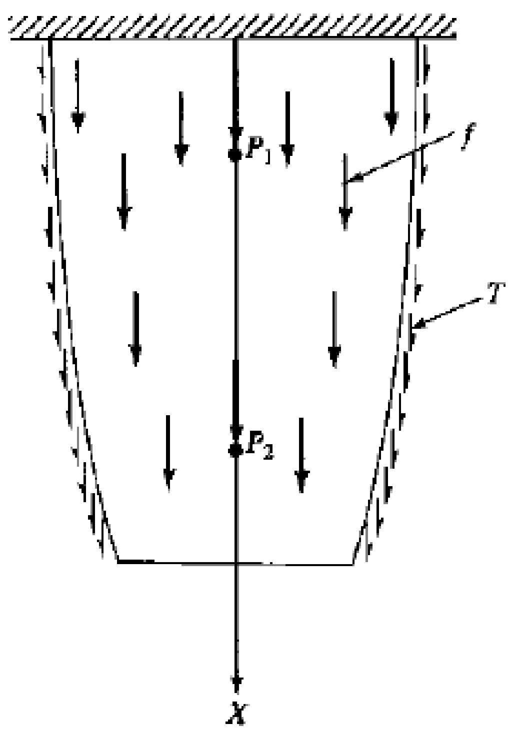


Figure 2.1 One-dimensional bar loaded by traction, body, and point loads.

Consider the bar in Fig. 2.1. The rust step is to model the bar as a stepped shaft, consisting of a discrete number of elements, each having a unifonn cross section. Specifically, let us model the bar using four finite elements. A simple scheme for doing this is to divide the bar into four regions, as shown in FIg. 3.2a. The average cross-sectional area within each region is evaluated and then used to define an element with uniform cross section. The resulting four-element, five-node finite element model is shown in Fig. 2.2b. In the finite element model, every element connects to two nodes. In Fig. 2.2b, the element numbers are circled to distinguish them from node numbers. In addition to the cross section, traction and body forces are also (normally) treated as constant within each element. However, cross-sectional area, traction, and body forces can differ in magnitude from element to element. Better approximations are obtained by increasing the number of elements. It is convenient to define a node at each location where a point load is applied.

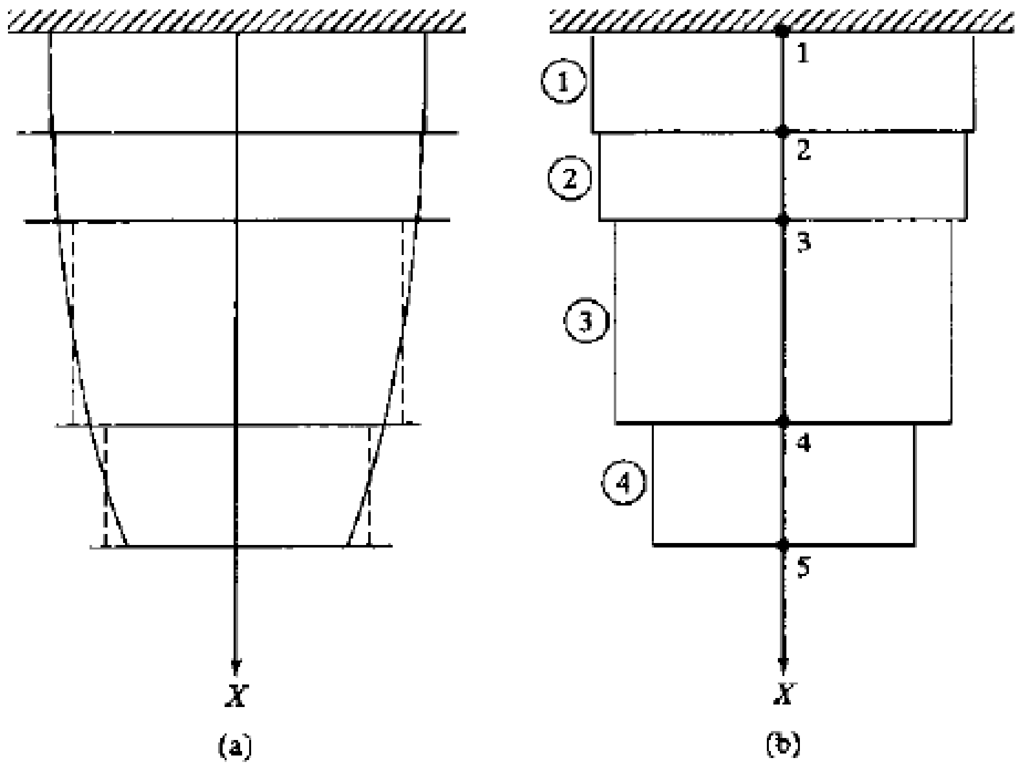
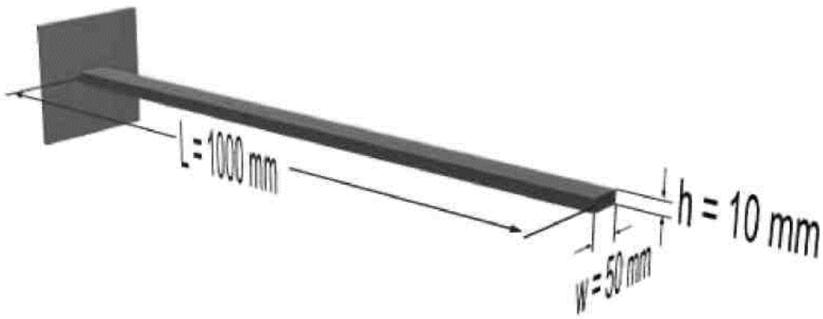


Figure 2.2. Finite element modeling of a bar.

5. Applications/ Simulation/ related Laboratory example

Lab Name: ME7P8/ Simulation & Analysis Laboratory

1. Find the maximum deflection caused by the weight of the beam itself. This beam is to be made of steel with a modulus of elasticity of 200 GPa. ρ=7.86×10-6 kg/mm^3



6. MCQ-Post test

1. Finite element analysis deals with
   (a) approximate numerical solution
   (b) non-boundary value problems
   (c) partial differential equations
   (d) laplace equations

2. FEM also operates the parameters like
   (a) heat transfer
   (b) temperature
   (c) Potential
   (d) All of the above

3. In one dimensional, the stress and strain relation is given by
   (a) $\sigma = E \in$
   (b) $\sigma = E / \in$
   (c) $\sigma = \in / E$
   (d) $\sigma = E - \in$

7. Conclusion

The one dimensional problem of finite element modeling is discussed.

8. References

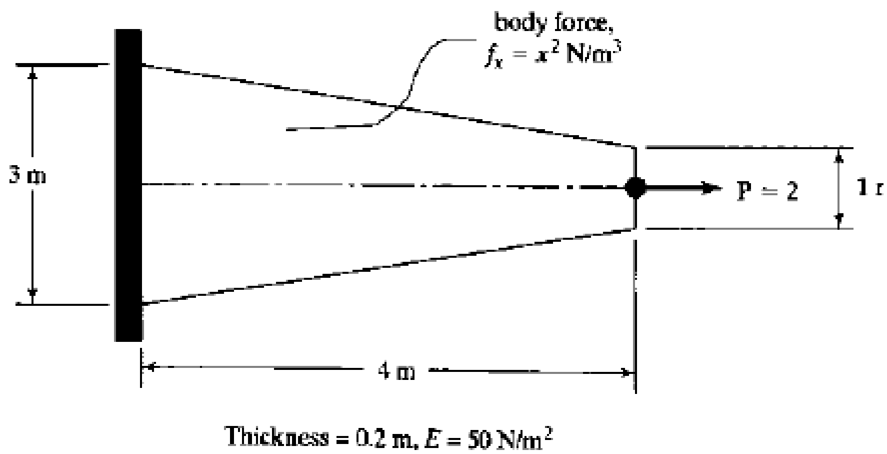
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Audio/Video-If any

https://www.youtube.com/watch?v=C6X9Ry02mPU

10.Assignments

1. For a given taper bar, do the finite element modelling with suitable number of elements



UNIT-2

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **One Dimensional Problems**

Name of the Topic : **Coordinates and shape functions**

1. Aim and Objectives

To understand the Coordinates and shape functions for one dimensional problems

2. Pre-Test-MCQ type

1. Finite element analysis is not used for
   (a) Complex problem solution
   (b) Non-homogeneous material solution
   (c) Anisotropic material solution
   (d) Exact solution

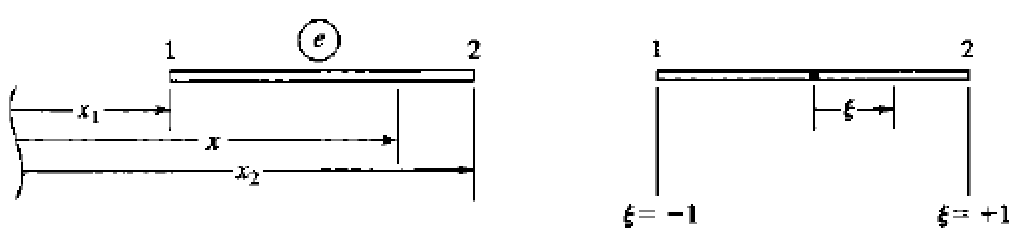
2. Finite element is -------------------
   (a) Small unit having definite shape and nodes
   (b) Small unit having definite shape and no nodes
   (c) Small unit only
   (d) Only nodes

3. Prerequisites

The realize the concept of Coordinates and shape functions for 1D element

4. Theory behind – Coordinates and shape functions

Consider a typical finite element *e* in Fig. 2.3a. In the local number scheme, the first node will be numbered 1 and the second node 2. The notation x_1= *x*-coordinate of node 1, x_2=*x*-coordinate of node 2 is used. We define a natural or intrinsic coordinate system, denoted by ξ, as



(a) (b)

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1 \tag{3.4}$$

From Fig. 3.5b, we see that $\xi = -1$ at node 1 and $\xi = 1$ at node 2. The length of an element is covered when ξ changes from -1 to 1. We use this system of coordinates in defining shape functions, which are used in interpolating the displacement field.

Now *the unknown displacement field within an element will be interpolated by a linear distribution* (Fig. 3.6). This approximation becomes increasingly accurate as more elements are considered in the model. To implement this linear interpolation, linear **shape functions** will be introduced as

$$N_1(\xi) = \frac{1 - \xi}{2} \tag{3.5}$$

$$N_2(\xi) = \frac{1 + \xi}{2} \tag{3.6}$$

The shape functions N_1 and N_2 are shown in Figs. 3.7a and b, respectively. The graph of the shape function N_1 in Fig. 3.7a is obtained from Eq. 3.5 by noting that $N_1 = 1$ at $\xi = -1$, $N_1 = 0$ at $\xi = 1$, and N_1 is a straight line between the two points. Similarly, the graph of N_2 in Fig. 3.7b is obtained from Eq. 3.6. Once the shape functions are defined, the linear displacement field within the element can be written in terms of the nodal displacements q_1 and q_2 as

$$u = N_1 q_1 + N_2 q_2 \tag{3.7a}$$

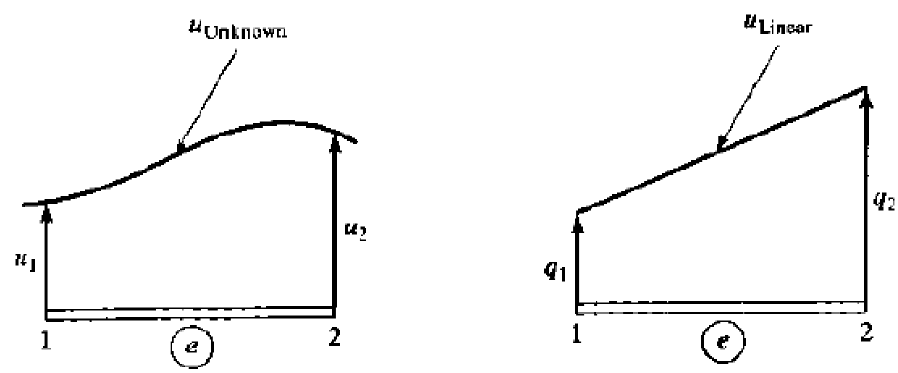


FIGURE 3.6 Linear interpolation of the displacement field within an element.

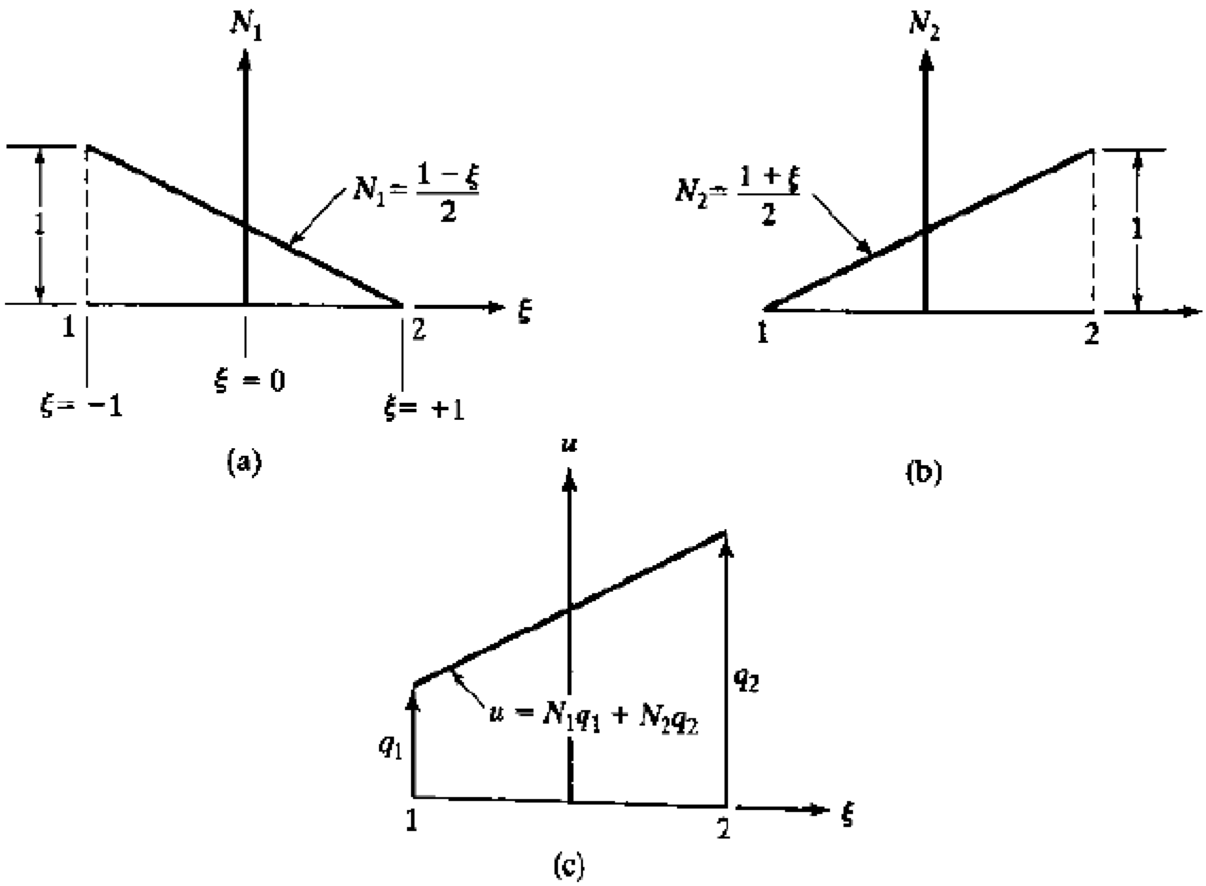


FIGURE 3.7 (a) Shape function N_1, (b) shape function N_2, and (c) linear interpolation using N_1 and N_2.

or, in matrix notation, as

$$u = \mathbf{Nq} \tag{3.7b}$$

where

$$\mathbf{N} = [N_1, N_2] \quad \text{and} \quad \mathbf{q} = [q_1, q_2]^T \tag{3.8}$$

In these equations, $\mathbf{q}$ is referred to as the *element displacement vector*. It is readily verified from Eq. 3.7a that $u = q_1$ at node 1, $u = q_2$ at node 2, and that u varies linearly (Fig. 3.7c).

It may be noted that the transformation from x to ξ in Eq. 3.4 can be written in terms of N_1 and N_2 as

$$x = N_1x_1 + N_2x_2 \tag{3.9}$$

5. Applications/ Simulation/ related Laboratory example

For all one dimensional problems the Coordinates and shape functions widely applicable.

6. MCQ-Post test

1. The points in the entire structure are defined using coordinates system is known as
   (a) Local coordinates
   (b) Natural coordinates
   (c) Global coordinate system
   (d) None of the above
2. The minimum number of dimensions are required to define the position of a point in space is:
   (a) one
   (b) two
   (c) three
   (d) four
3. Sum of shape functions =
   (a) 1
   (b) 2
   (c) 3
   (d) 0
4. Shape functions are called as
   (a) Shape size functions
   (b) FEM Functions
   (c) Interpolation functions
   (d) Meshing functions

7. Conclusion

The Coordinates and shape functions is constructively studied.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Audio/Video-If any

https://www.youtube.com/watch?v=rb4AOTm_tBA

10. Assignments

1. Plot the shape function of bar element with neat sketch.

UNIT-2

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **One Dimensional Problems**

Name of the Topic : **Potential energy approach and Galerkin approach**

1. Aim and Objectives

To study on Potential energy and Galerkin approaches for 1D problems

2. Pre-Test-MCQ type

1. Differentiate y = sec $(x^2 + 2)$
   (a) 2x cos $(x^2 + 2)$
   (b) –cos $(x^2 + 2)$ cot $(x^2 + 2)$
   (c) 2x sec $(x^2 + 2)$ tan $(x^2 + 2)$
   (d) cos $(x^2 +2)$

2. Differentiate $(x^2 + 2)1/2$

   (a) $((x^2 + 2)1/2) / 2$
   (b) $x / (x^2 + 2)1/2$
   (c) $(2x) / (x^2 + 2)1/2$
   (d) $(x^2 + 2)3/2$

3. Differentiate the equation $y = x^2 / (x +1)$

   (a) $(x^2 + 2x) / (x + 1)2$
   (b) x / (x + 1)
   (c) 2x
   (d) (2×2) / (x + 1)

3. Prerequisites

The proficiency knowledge on differential calculus is needed.

4. Theory behind – Potential energy approach and Galerkin approach

Potential energy approach

The general expression for the potential energy given in Chapter 1 is

$$\Pi = \frac{1}{2}\int_L \sigma^T \epsilon A\, dx - \int_L u^T f A\, dx - \int_L u^T T\, dx - \sum_i u_i P_i \qquad (3.17)$$

The quantities σ, ϵ, u, f, and T in Eq. 3.17 are discussed at the beginning of this chapter. In the last term, P_i represents a force acting at point i, and u_i is the x displacement at that point. The summation on i gives the potential energy due to all point loads.

Since the continuum has been discretized into finite elements, the expression for Π becomes

$$\Pi = \sum_e \frac{1}{2} \int_e \sigma^{\mathrm{T}} \epsilon A\, dx - \sum_e \int_e u^{\mathrm{T}} f A\, dx - \sum_e \int_e u^{\mathrm{T}} T\, dx - \sum_i Q_i P_i \qquad (3.18a)$$

The last term in Eq. 3.18a assumes that point loads P_i are applied at the nodes. This assumption makes the present derivation simpler with respect to notation and is also a common modeling practice. Equation 3.18a can be written as

$$\Pi = \sum_e U_e - \sum_e \int_e u^{\mathrm{T}} f A\, dx - \sum_e \int_e u^{\mathrm{T}} T\, dx - \sum_i Q_i P_i \qquad (3.18b)$$

where

$$U_e = \frac{1}{2} \int \sigma^{\mathrm{T}} \epsilon A\, dx$$

is the element strain energy.

Galerkin's approach

Following the concepts introduced in Chapter 1, we introduce a virtual displacement field

$$\phi = \phi(x) \qquad (3.37)$$

and associated virtual strain

$$\epsilon(\phi) = \frac{d\phi}{dx} \qquad (3.38)$$

where ϕ is an arbitrary or virtual displacement consistent with the boundary conditions. Galerkin's variational form, given in Eq. 1.43, for the one-dimensional problem considered here, is

$$\int_L \sigma^{\mathrm{T}} \epsilon(\phi) A\, dx - \int_L \phi^{\mathrm{T}} f A\, dx - \int_L \phi^{\mathrm{T}} T\, dx - \sum_i \phi_i P_i = 0 \qquad (3.39a)$$

This equation should hold for every ϕ consistent with the boundary conditions. The first term represents the internal virtual work, while the load terms represent the external virtual work.

On the discretized region, Eq. 3.39a becomes

$$\sum_e \int_e \epsilon^{\mathrm{T}} E \epsilon(\phi) A\, dx - \sum_e \int_e \phi^{\mathrm{T}} f A\, dx - \sum_e \int_e \phi^{\mathrm{T}} T\, dx - \sum_i \phi_i P_i = 0 \qquad (3.39b)$$

Note that ϵ is the strain due to the actual loads in the problem, while $\epsilon(\phi)$ is a virtual strain. Similar to the interpolation steps in Eqs. 3.7b, 3.14, and 3.16, we express

$$\phi = \mathbf{N}\psi$$

$$\epsilon(\phi) = \mathbf{B}\psi \tag{3.40}$$

where $\psi = [\psi_1, \psi_2]^T$ represents the arbitrary nodal displacements of element e. Also, the global virtual displacements at the nodes are represented by

$$\mathbf{\Psi} = [\psi_1, \psi_2, \ldots, \psi_N]^T \tag{3.41}$$

5. Applications/ Simulation/ related Laboratory example

To Potential energy and Galerkin approaches is used extensively used in 1D problems

6. MCQ-Post test

1. The value of ϕ is equal to
   (a) $\phi < N\psi$
   (b) $\phi > N\psi$
   (c) $\phi = N\psi$
   (d) None of the above
2. The expression of strain energy U_e

(a) $U_e = \frac{1}{2}\int \sigma^T A^2 dx$

(b) $U_e = \frac{1}{2}\int \sigma\varepsilon A dx$

(c) $U_e = \frac{1}{2}\int \sigma^T \varepsilon A dx$

(d)one of the above

7. Conclusion

The Potential energy and Galerkin approaches were discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON “Fundamentals of Finite Element Analysis”2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- RAO S.S., “The Finite Element Method in Engineering”, Pergammon Press, 1989
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=sOvHM-L7e_Q

10. Assignments

1. Briefly explain about Potential energy approach with examples.
2. Briefly explain about Galerkin approach with examples.

UNIT-2

Name of the Course	:	**FINITE ELEMENT ANALYSIS (FEA)**
Name of the Unit	:	**One Dimensional Problems**
Name of the Topic	:	**Assembly of stiffness matrix and load vector – Finite element equations**

1. Aim and Objectives

- To understand the Assembly of stiffness matrix and load vector
- To make out the knowledge on finite element equations

2. Pre-Test-MCQ type

1. The energy possessed by a body due to its position is called its
   (a) heat energy
   (b) kinetic energy
   (c) potential energy
   (d) chemical energy

2. When a body is lifted through a height h, the work done on it appears in the form of its
   (a) kinetic energy
   (b) potential energy
   (c) chemical energy
   (d) geothermal energy

3. The energy present in a body due to its height is called
   (a) gravitational kinetic energy
   (b) gravitational potential energy
   (c) altitude energy
   (d) gravitational energy

4. If A and B be real symmetric matrices of sizen n x n, then
   (a) $AA^T = 1$
   (b) $A = A^{-1}$
   (c) $AB = BA$
   (d) $(AB)^T = BA$

3. Prerequisites

The comprehension knowledge of fundamentals of engineering science is required.

4. Theory behind

Assembly of stiffness matrix and load vector

3.6 ASSEMBLY OF THE GLOBAL STIFFNESS MATRIX AND LOAD VECTOR

We noted earlier that the total potential energy written in the form

$$\Pi = \sum_e \frac{1}{2}\mathbf{q}^T\mathbf{k}^e\mathbf{q} - \sum_e \mathbf{q}^T\mathbf{f}^e - \sum_e \mathbf{q}^T\mathbf{T}^e - \sum_i P_iQ_i$$

can be written in the form

$$\Pi = \tfrac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} - \mathbf{Q}^T\mathbf{F}$$

by taking element connectivity into account. This step involves assembling $\mathbf{K}$ and $\mathbf{F}$ from element stiffness and force matrices. The assembly of the structural stiffness matrix $\mathbf{K}$ from element stiffness matrices $\mathbf{k}^e$ will first be shown here.

Referring to the finite element model in Fig. 3.2b, let us consider the strain energy in, say, element 3. We have

$$U_3 = \tfrac{1}{2}\mathbf{q}^T\mathbf{k}^3\mathbf{q} \tag{3.52a}$$

or, substituting for $\mathbf{k}^3$,

$$U_3 = \frac{1}{2}\mathbf{q}^T\frac{E_3A_3}{\ell_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{q} \tag{3.52b}$$

For element 3, we have $\mathbf{q} = [Q_3, Q_4]^T$. Thus, we can write U_3 as

$$U_3 = \tfrac{1}{2}[Q_1, Q_2, Q_3, Q_4, Q_5]\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{E_3A_3}{\ell_3} & \frac{-E_3A_3}{\ell_3} & 0 \\ 0 & 0 & \frac{-E_3A_3}{\ell_3} & \frac{E_3A_3}{\ell_3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} \tag{3.53}$$

From the previous equations, we see that elements of the matrix $\mathbf{k}^3$ occupy the third and fourth rows and columns of the $\mathbf{K}$ matrix. Consequently, when adding element-strain energies, the elements of $\mathbf{k}^e$ are placed in the appropriate locations of the global $\mathbf{K}$ matrix, based on the element connectivity; overlapping elements are simply added. We can denote this assembly symbolically as

$$\mathbf{K} \leftarrow \sum_e \mathbf{k}^e \tag{3.54a}$$

Similarly, the global load vector $\mathbf{F}$ is assembled from element-force vectors and point loads as

The Galerkin approach also gives us the same assembly procedure. An example is now given to illustrate this assembly procedure in detail. In actual computation, **K** is stored in banded or skyline form to take advantage of symmetry and sparsity. This aspect is discussed in Section 3.7 and in greater detail in Chapter 4.

Sample problem

Consider the bar as shown in Fig. E3.2. For each element i, A_i and ℓ_i are the cross-sectional area and length, respectively. Each element i is subjected to a traction force T_i per unit length and a body force f per unit volume. The units of T_i, f, A_i, and so on are assumed to be consistent. The Young's modulus of the material is E. A concentrated load P_2 is applied at node 2. The structural stiffness matrix and nodal load vector will now be assembled.

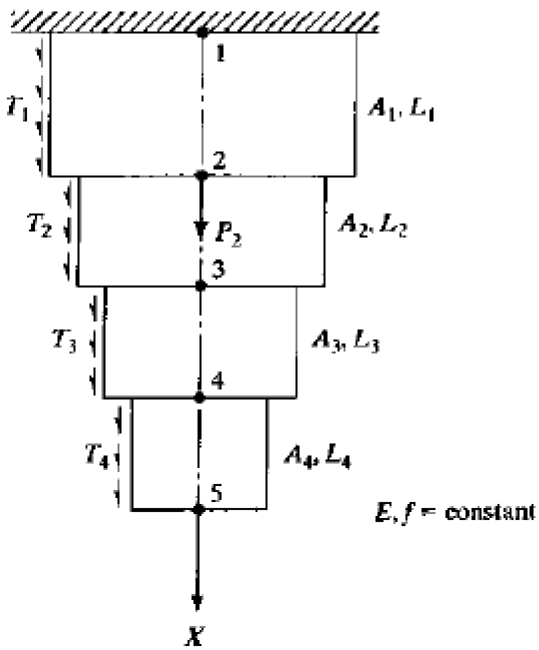


FIGURE E3.2

The element stiffness matrix for each element i is obtained from Eq. 3.26 as

$$[k^{(i)}] = \frac{EA_i}{\ell_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The element connectivity table is the following:

Element	1	2
1	1	2
2	2	3
3	3	4
4	4	5

The element stiffness matrices can be "expanded" using the connectivity table and then summed (or assembled) to obtain the structural stiffness matrix as follows:*

$$\mathbf{K} = \frac{EA_1}{\ell_1}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA_2}{\ell_2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{EA_3}{\ell_3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA_4}{\ell_4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

which gives

$$\mathbf{K} = E\begin{bmatrix} \frac{A_1}{\ell_1} & -\frac{A_1}{\ell_1} & 0 & 0 & 0 \\ -\frac{A_1}{\ell_1} & \left(\frac{A_1}{\ell_1} + \frac{A_2}{\ell_2}\right) & -\frac{A_2}{\ell_2} & 0 & 0 \\ 0 & -\frac{A_2}{\ell_2} & \left(\frac{A_2}{\ell_2} + \frac{A_3}{\ell_3}\right) & -\frac{A_3}{\ell_3} & 0 \\ 0 & 0 & -\frac{A_3}{\ell_3} & \left(\frac{A_3}{\ell_3} + \frac{A_4}{\ell_4}\right) & -\frac{A_4}{\ell_4} \\ 0 & 0 & 0 & -\frac{A_4}{\ell_4} & \frac{A_4}{\ell_4} \end{bmatrix}$$

The global load vector is assembled as

$$\mathbf{F} = \begin{Bmatrix} \dfrac{A_1\ell_1 f}{2} + \dfrac{\ell_1 T_1}{2} \\ \left(\dfrac{A_1\ell_1 f}{2} + \dfrac{\ell_1 T_1}{2}\right) + \left(\dfrac{A_2\ell_2 f}{2} + \dfrac{\ell_2 T_2}{2}\right) \\ \left(\dfrac{A_2\ell_2 f}{2} + \dfrac{\ell_2 T_2}{2}\right) + \left(\dfrac{A_3\ell_3 f}{2} + \dfrac{\ell_3 T_3}{2}\right) \\ \left(\dfrac{A_3\ell_3 f}{2} + \dfrac{\ell_3 T_3}{2}\right) + \left(\dfrac{A_4\ell_4 f}{2} + \dfrac{\ell_4 T_4}{2}\right) \\ \dfrac{A_4\ell_4 f}{2} + \dfrac{\ell_4 T_4}{2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ P_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Finite element equations

THE FINITE ELEMENT EQUATIONS; TREATMENT OF BOUNDARY CONDITIONS

Finite element equations are now developed after a consistent treatment of the boundary conditions.

Types of Boundary Conditions

After using a discretization scheme to model the continuum, we have obtained an expression for the total potential energy in the body as

$$\Pi = \tfrac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\mathbf{F}$$

where **K** is the structural stiffness matrix, **F** is the global load vector, and **Q** is the global displacement vector. As discussed previously, **K** and **F** are assembled from element stiffness and force matrices, respectively. We now must arrive at the equations of equilibrium, from which we can determine nodal displacements, element stresses, and support reactions.

The minimum potential-energy theorem (Chapter 1) is now invoked. This theorem is stated as follows: *Of all possible displacements that satisfy the boundary conditions of a structural system, those corresponding to equilibrium configurations make the total potential energy assume a minimum value.* Consequently, the equations of equilibrium can be obtained by minimizing, with respect to **Q**, the potential energy $\Pi = \frac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\mathbf{F}$ *subject to boundary conditions.* Boundary conditions are usually of the type

$$Q_{p_1} = a_1, Q_{p_2} = a_2, \ldots, Q_{p_r} = a_r \qquad (3.56)$$

That is, the displacements along dofs $p_1, p_2, \ldots, p_r$ are specified to be equal to a_1, $a_2, \ldots, a_r$, respectively. In other words, there are r number of supports in the structure, with each support node given a specified displacement. For example, consider the bar in Fig. 3.2b. There is only one boundary condition in this problem, $Q_1 = 0$.

It is noted here that *the treatment of boundary conditions in this section is applicable to two- and three-dimensional problems as well.* For this reason, the term dof is used here instead of node, since a two-dimensional stress problem will have two degrees of freedom per node. The steps described in this section will be used in all subsequent chapters. Furthermore, a Galerkin-based argument leads to the same steps for handling boundary conditions as the energy approach used subsequently.

There are *multipoint constraints* of the type

$$\beta_1 Q_{p_1} + \beta_2 Q_{p_2} = \beta_0 \tag{3.57}$$

where β_0, β_1, and β_2 are known constants. These types of boundary conditions are used in modeling inclined roller supports, rigid connections, or shrink fits.

5. Applications/ Simulation/ related Laboratory example

The assembly of stiffness matrix and load vector and associated finite element equations for applying the boundary conditions are more useful while solving all one dimensional problems.

6. MCQ-Post test

1. Which one of the following has the main property of a stiffness matrix?
   (a) The sum of elements in any column must be equal to zero
   (b) The sum of elements in any column must be not equal to zero
   (c) The sum of elements in any column must be equal to zero
   (d) None of these
2. In a particular axial deformation of bar problem, if one end is subjected by an axial load and it is specified, then the type of boundary condition is
   (a) Natural type
   (b) Mixed type
   (c) Essential type
   (d) Cauchy's type
3. A bar is modelled as 1-D element only if its
   (a) area of cross section is small
   (b) M.I is small
   (c) length is very large compared to cross sectional area
   (d) all of the above
4. Stiffness matrix contains information on
   (a) geometry
   (b) material properties
   (c) both
   (d) none

7. Conclusion

The assembly of stiffness matrix and load vector and finite element equations are discussed.

8. References

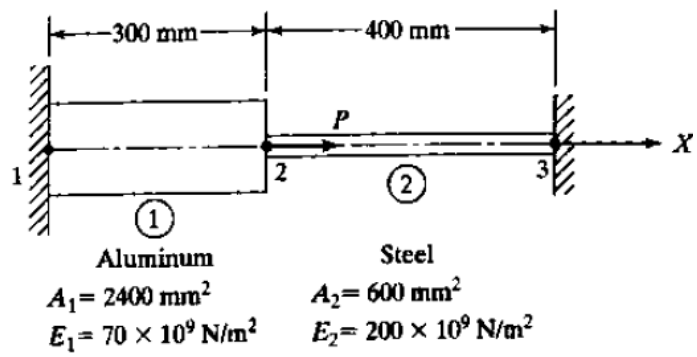
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON “Fundamentals of Finite Element Analysis”2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- RAO S.S., “The Finite Element Method in Engineering”, Pergammon Press, 1989
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=e6FKKR2hwLc
https://www.youtube.com/watch?v=aU8SScEGneE

10. Assignments

1. Consider the bar shown in Figure. An axial load $P=200\times10^3$N is applied as shown. Find the nodal Displacements, stress in each material and reactions forces.



UNIT-2

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **One Dimensional Problems**

Name of the Topic : **Quadratic shape functions**

1. Aim and Objectives

To know the knowledge on development of Quadratic shape functions

2. Pre-Test-MCQ type

1. The order of the differential equation $2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ *is*

   (a) 2
   (b) 1
   (c) 0
   (d) not defined

2. The solution of the differential equation $x^2 + y^2\frac{dy}{dx} = 4$ *is*

   (a) $x^2 + y^2 = 12x + c$
   (b) $x^2 + y^2 = 3x + c$
   (c) $x^3 + y^3 = 3x + c$
   (d) $x^3 + y^3 = 12x + c$

3. Prerequisites

The knowledge of differential calculus required.

4. Theory behind

So far, the unknown displacement field was interpolated by linear shape functions within each element. In some problems, however, use of quadratic interpolation leads to far more accurate results. In this section, quadratic shape functions will be introduced, and the corresponding element stiffness matrix and load vectors will be derived. The reader should note that the basic procedure is the same as that used in the linear one-dimensional element earlier.

Consider a typical three-node quadratic element, as shown in Fig. 3.11a. In the local numbering scheme, the left node will be numbered 1, the right node 2, and the midpoint 3. Node 3 has been introduced for the purposes of passing a quadratic fit and is called an internal node. The notation x_i = x-coordinate of node *i, j*= 1, 2, 3, is used. Further, $q = [q_1, q_2, q_3]^T$, where q1,q2 and q3 are the displacements of nodes 1,2,and 3, respectively. The x-coordinate system is mapped onto a ξ-coordinate system, which is given by the transformation

$$\xi = \frac{2(x - x_3)}{x_2 - x_1} \tag{3.86}$$

From Eq. 3.86, we see that $\xi = -1, 0$, and $+1$ at nodes 1, 3, and 2 (Fig. 3.11b).

Now, in ξ-coordinates, *quadratic shape functions* N_1, N_2, and N_3 will be introduced as

$$N_1(\xi) = -\tfrac{1}{2}\xi(1 - \xi) \tag{3.87a}$$

$$N_2(\xi) = \tfrac{1}{2}\xi(1 + \xi) \tag{3.87b}$$

$$N_3(\xi) = (1 + \xi)(1 - \xi) \tag{3.87c}$$

The shape function N_1 is equal to unity at node 1 and zero at nodes 2 and 3. Similarly, N_2 equals unity at node 2 and equals zero at the other two nodes; N_3 equals unity at node 3 and equals zero at nodes 1 and 2. The shape functions N_1, N_2, and N_3 are graphed in Fig. 3.12. The expressions for these shape functions can be written down by inspection. For example, since $N_1 = 0$ at $\xi = 0$ and $N_1 = 0$ at $\xi = 1$, we know that N_1 must contain the product $\xi(1 - \xi)$. That is, N_1 is of the form

$$N_1 = c\xi(1 - \xi) \tag{3.88}$$

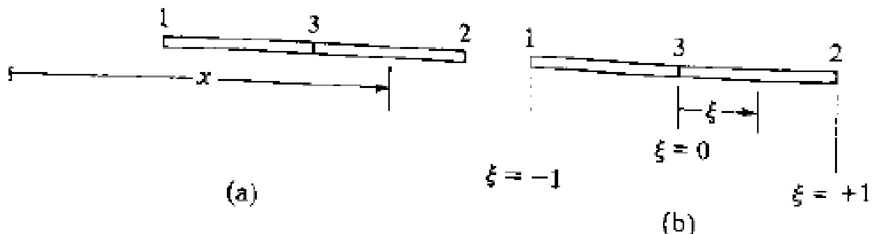


FIGURE 3.11 Quadratic element in x- and ξ-coordinates.

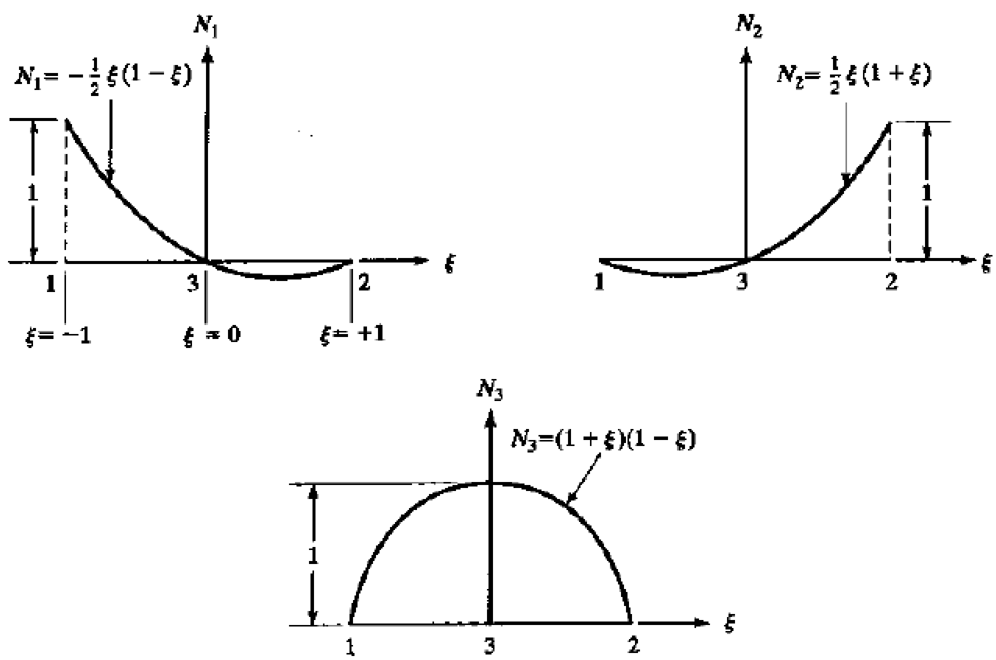


FIGURE 3.12 Shape functions N_1, N_2, and N_3.

The constant c is now obtained from the condition $N_1 = 1$ at $\xi = -1$, which yields $c = -\frac{1}{2}$, resulting in the formula given in Eq. 3.87a. These shape functions are called *Lagrange* shape functions.

Now the displacement field within the element is written in terms of the nodal displacements as

$$u = N_1 q_1 + N_2 q_2 + N_3 q_3 \tag{3.89a}$$

or

$$u = \mathbf{N}\mathbf{q} \tag{3.89b}$$

where $\mathbf{N} = [N_1, N_2, N_3]$ is a (1×3) vector of shape functions and $\mathbf{q} = [q_1, q_2, q_3]^T$ is the (3×1) element displacement vector. At node 1, we see that $N_1 = 1, N_2 = N_3 = 0$, and hence $u = q_1$. Similarly, $u = q_2$ at node 2 and $u = q_3$ at node 3. Thus, u in Eq. 3.89a is a quadratic interpolation passing through q_1, q_2, and q_3 (Fig. 3.13).
The strain ϵ is now given by

$$\begin{aligned}
\epsilon &= \frac{du}{dx} && \text{(strain–displacement relation)} \\
&= \frac{du}{d\xi}\frac{d\xi}{dx} && \text{(chain rule)} \\
&= \frac{2}{x_2 - x_1}\frac{du}{d\xi} && \text{(using Eq. 3.86)} \\
&= \frac{2}{x_2 - x_1}\left[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}\right]\cdot\mathbf{q} && \text{(using Eq. 3.89)}
\end{aligned} \tag{3.90}$$

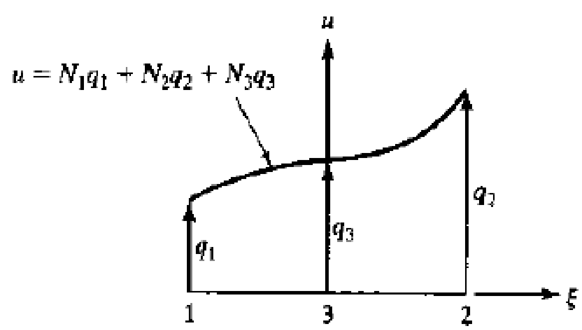


FIGURE 3.13 Interpolation using quadratic shape functions.

Using Eqs. 3.87, we have

$$\epsilon = \frac{2}{x_2 - x_1}\left[-\frac{1 - 2\xi}{2}, \frac{1 + 2\xi}{2}, -2\xi\right]\mathbf{q} \tag{3.91}$$

which is of the form

$$\epsilon = \mathbf{B}\mathbf{q} \tag{3.92}$$

where **B** is given by

$$\mathbf{B} = \frac{2}{x_2 - x_1}\left[-\frac{1-2\xi}{2}, \frac{1+2\xi}{2}, -2\xi\right] \tag{3.93}$$

Using Hooke's law, we can write the stress as

$$\sigma = E\,\mathbf{Bq} \tag{3.94}$$

5. Applications/ Simulation/ related Laboratory example

The quadratic bar element is mainly used for complex profile of 1D problems

6. MCQ-Post test

1. The characteristics of the shape functions is/are
   - (a) the shape function has unit value at one nodal point and zero value at the other nodes
   - (b) the sum of the shape function is equal to one
   - **(c) a & b**
   - (d) none
2. Primary variable in FEM structural analysis is
   - (a) force
   - **(b) Displacement**
   - (c) Stress
   - (d) Strain

3. Each node of a quadratic 1-D beam element has __________ degrees of freedom
   - a. 2
   - b. 1
   - c. 4
   - **d. 3**

4. Why polynomial type of interpolation functions are mostly used in FEM analysis
   - (a) It is easy to formulate and computerize the finite element equations
   - (b) It is easy to perform differential or integration
   - (c) The accuracy of the results can be improved by increasing the order of the polynomial
   - **(d) All of these**

7. Conclusion

The quadratic shape function of 3 noded 1D bar element is discussed.

8. References

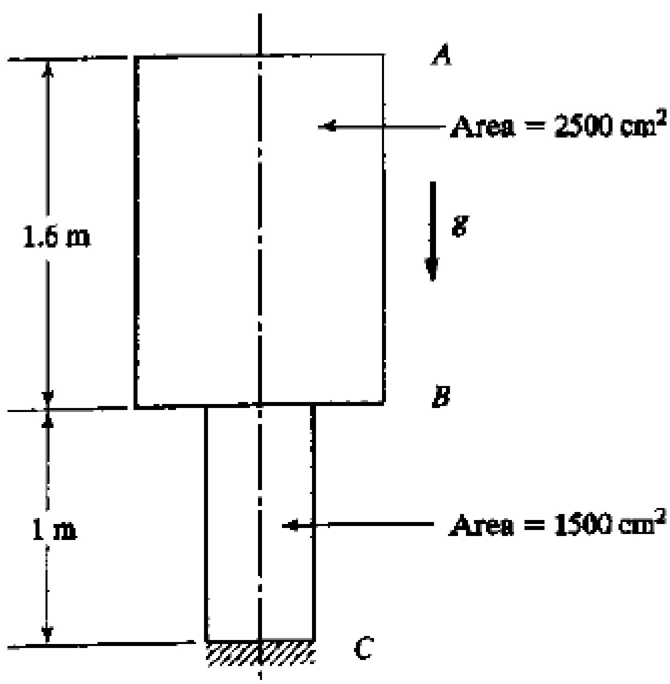
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=XYbyuaYVQb8

10. Assignments

1. For the vertical rod shown in Figure, find the deflection at *A* and the stress distribution. Use E= l00MPa and weight per unit volume = 0.06N/cm^2 Comment on the stress distribution.



UNIT-2

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **One Dimensional Problems**

Name of the Topic : **Applications to plane trusses**

1. Aim and Objectives

To expertise in finite element analysis of plane truss analysis

2. Pre-Test-MCQ type

1. The finite element methods can be applied in _____________areas.
   (a) Thermal
   (b) Soil and rock mechanics
   (c) Noise Problems
   (d) All
2. Determinant of assembled stiffness matrix before applying boundary conditions is
   (a) < 0
   (b) = 0
   (c) 0
   (d) depends on the problem
3. ________ is/are the phase/s of finite element method
   (a) Preprocessing
   (b) Solution
   (c) Post Processing
   (d) All of these

3. Prerequisites

The basics of finite element analysis is required.

4. Theory behind- Applications to plane trusses

The finite element analysis of truss structures is presented in this chapter. A typical plane truss is shown in Fig. 4.1. A truss structure consists only of two-force members. That is, every truss element is in direct tension or compression (Fig. 4.2). In a truss, it is required that all loads and reactions are applied only at the joints and that all members are connected together at their ends by frictionless pin joints. Every engineering student has, in a course on statics. analyzed trusses using the method of joints and the method of sections. These methods. while illustrating the fundamentals of statics. become tedious when applied to large-scale statically indeterminate truss structures. Further, joint displacements are not readily obtainable. The finite element method on the other hand is applicable to statically determinate or indeterminate structures alike. The finite element method also provides joint deflections. Effects of temperature changes and support settlements can also be routinely bandied.

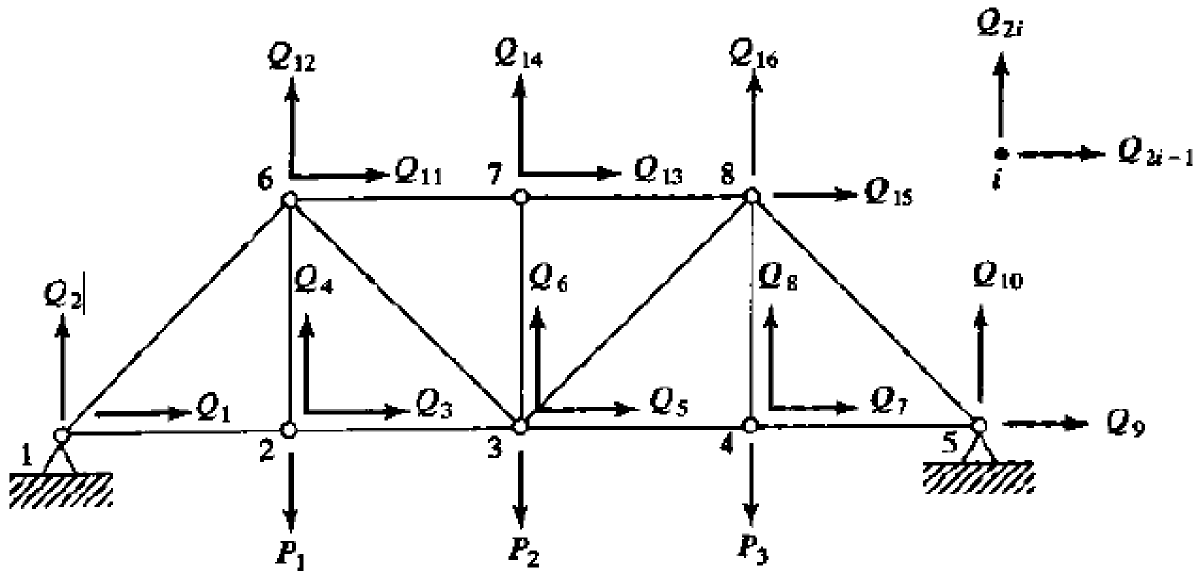


FIGURE 4.1 A two-dimensional truss.

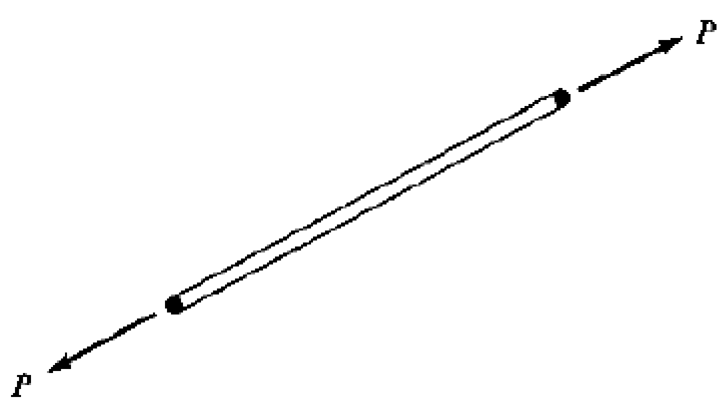


FIGURE 4.2 A two-force member.

A typical plane-truss element is shown in local and global coordinate systems in Fig. 4.3. In the local numbering scheme, the two nodes of the element are numbered 1 and 2. The local coordinate system consists of the *x'* -axis, which runs along the element from node 1 toward node 2. All quantities in the local coordinate system will be denoted by a prime (').The lobal x-,y-coordinate system is fixed and does not depend on the orientation of the element. Note that *x,y,* and *z* form a right -handed coordinate system with the z-axis coming straight out of the paper. In the global coordinate system every node has two degrees of freedom (dofs).A systematic numbering scheme is adopted here: A node whose global node number is *j* has associated with it dofs $2j$ - 1 and $2j$. Further, the global displacements associated with node j are Q_{2i-1} and Q_{2j}, as shown in Fig. 4.1.

Let q'_1 and q'_2 be the displacements of nodes 1 and 2, respectively, in the local coordinate system. Thus, the element displacement vector in the local coordinate system is denoted by

$$\mathbf{q}' = [q'_1, q'_2]^T \tag{4.1}$$

The element displacement vector in the global coordinate system is a (4 × 1) vector denoted by

$$\mathbf{q} = [q_1, q_2, q_3, q_4]^T \tag{4.2}$$

The relationship between $\mathbf{q}'$ and $\mathbf{q}$ is developed as follows: In Fig. 4.3b, we see that q'_1 equals the sum of the projections of q_1 and q_2 onto the x'-axis. Thus,

$$q'_1 = q_1 \cos\theta + q_2 \sin\theta \tag{4.3a}$$

Similarly,

$$q'_2 = q_3 \cos\theta + q_4 \sin\theta \tag{4.3b}$$

At this stage, the **direction cosines** ℓ and m are introduced as $\ell = \cos\theta$ and $m = \cos\phi$ ($= \sin\theta$). These direction cosines are the cosines of the angles that the local x'-axis makes with the global x-, y-axes, respectively. Equations 4.3a and 4.3b can now be written in matrix form as

$$\mathbf{q}' = \mathbf{L}\mathbf{q} \tag{4.4}$$

where the **transformation matrix L** is given by

$$\mathbf{L} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix} \tag{4.5}$$

Formulas for Calculating ℓ and m

Simple formulas are now given for calculating the direction cosines ℓ and m from nodal coordinate data. Referring to Fig. 4.4, let (x_1, y_1) and (x_2, y_2) be the coordinates of nodes 1 and 2, respectively. We then have

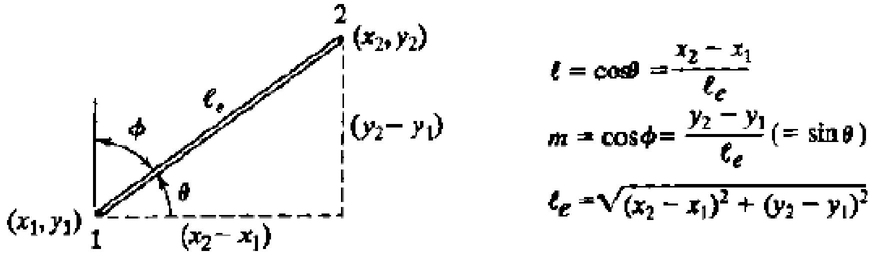


FIGURE 4.4 Direction cosines.

$$\ell = \frac{x_2 - x_1}{\ell_e} \qquad m = \frac{y_2 - y_1}{\ell_e} \tag{4.6}$$

where the length ℓ_e is obtained from

$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.7}$$

Element Stiffness Matrix

An important observation will now be made: *The truss element is a one-dimensional element when viewed in the local coordinate system.* This observation allows us to use previously developed results in Chapter 3 for one-dimensional elements. Consequently, from Eq. 3.26, the element stiffness matrix for a truss element in the local coordinate system is given by

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.8}$$

where A_e is the element cross-sectional area and E_e is Young's modulus. The problem at hand is to develop an expression for the element stiffness matrix in the global coordinate system. This is obtainable by considering the strain energy in the element. Specifically, the element strain energy in local coordinates is given by

$$U_e = \tfrac{1}{2}\mathbf{q}'^{\mathrm{T}}\mathbf{k}'\mathbf{q}' \tag{4.9}$$

Substituting for $\mathbf{q}' = \mathbf{L}\mathbf{q}$ into Eq. 4.9, we get

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}[\mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}]\mathbf{q} \tag{4.10}$$

The strain energy in global coordinates can be written as

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}\mathbf{q} \tag{4.11}$$

where $\mathbf{k}$ is the element stiffness matrix in global coordinates. From the previous equation, we obtain the element stiffness matrix in global coordinates as

$$\mathbf{k} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L} \tag{4.12}$$

Substituting for $\mathbf{L}$ from Eq. 4.5 and for $\mathbf{k}'$ from Eq. 4.8, we get

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix} \tag{4.13}$$

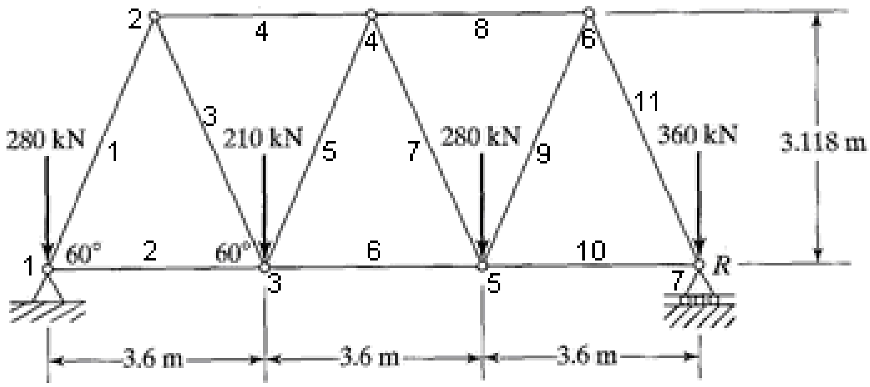
The element stiffness matrices are assembled in the usual manner to obtain the structural stiffness matrix. This assembly is illustrated in Example 4.1. The computer logic for directly placing element stiffness matrices into global matrices for banded and skyline solutions is explained in Section 4.4.

5. Applications/ Simulation/ related Laboratory example

The main application of plan truss element used in mechanical truss problems.

Lab Name: ME7P8/ Simulation & Analysis Laboratory

Determine the nodal deflections, reaction forces, and stress for the truss system shown below (E = 200GPa, A = 3250mm^2).



6. MCQ-Post test

1. __________ is a structure made of slender members which are joined together at their end points.
   (a) Truss
   (b) Beam
   (c) Pillar
   (d) Support
2. _________ trusses lie on a plane.
   (a) Planar
   (b) 2D
   (c) Linear
   (d) 3D

3. To design the trusses which of the following rules is followed?
   (a) All the loads are applied by the use of cables
   (b) The loads are applied at the joints
   (c) All the loads are not applied at the joints
   (d) The loads are not applied at all to the joints

4. In truss analysis, the reactions can be found by using the equation ______.
   (a) R=KQ+F
   (b) R=KQ-F
   (c) R=K+QF
   (d) R=K-Q

7. Conclusion

The application of truss element suitably applied in plane truss problems.

8. References

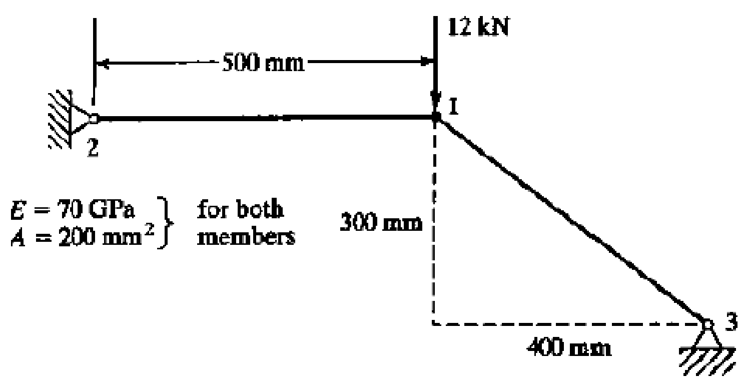
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

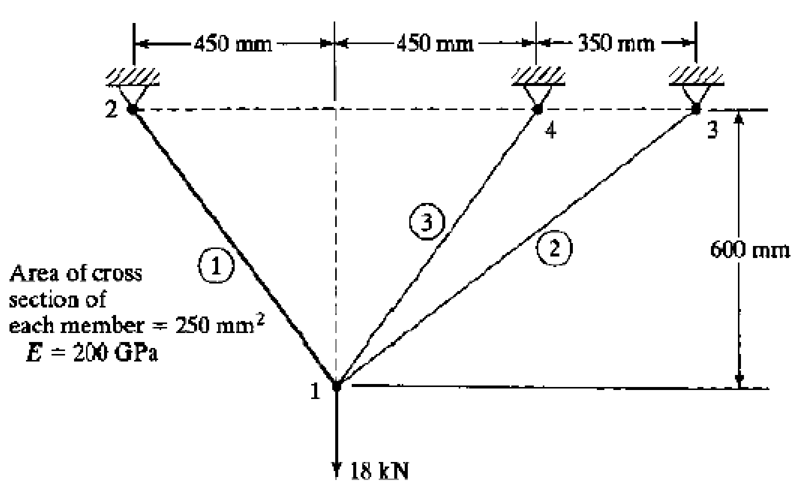
https://www.youtube.com/watch?v=m5Ng0C5ZFJ8

10.Assignments

1. For the two-bar truss shown in Figure, determine the displacements of node 1 and the stress in element 1-3.



2.For the three-bar truss shown in Fig. P4.7, detennine the displacements of node I and the stress in element 3.



UNIT-3

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Two Dimensional Continuum**

Name of the Topic : **Introduction – Finite element modelling**

1. Aim and Objectives

To understand the Finite element modeling of Two Dimensional Continuum

2. Pre-Test-MCQ type

1. Number of displacement polynomials used for an element depends on
   (a) Nature of element
   (b) Type of an element
   (c) Degrees of freedom
   (d) Nodes
2. At fixed support, the displacements are equal to
   (a) 1
   (b) 2
   (c) 3
   (d) 0
3. What is the traction force of a 2D body?
   (a) Force per unit area
   (b) force per unit length
   (c) force per unit volume
   (d) none of these

3. Prerequisites

- The vital information of one dimensional FEA problems is required.
- The basics of engineering mechanics is required.

4. Theory behind

Introduction

The two-dimensional finite element formulation in this chapter follows the steps used in the one-dimensional problem. The displacements, traction components, and distributed body force values are functions of the position indicated by *(x, y).* The displacement vector *u* is given as

$$\boldsymbol{u}=[u,v]^{T}$$

where *u* and *v* are the *x* and y components of *u*, respectively. The stresses and strains are given by

$\sigma=[\ \sigma_x, \sigma_y, \sigma_z]^T$

$\varepsilon=[\ \varepsilon_x, \varepsilon_y, \varepsilon_z]^T$

From Fig 5.1, representing the two-dimensional problem in a general setting. the body force, traction vector, and elemental volume are given by

$$f=[f_x, f_y]^T \quad T=\left[T_x, T_y\right]^T \quad and \ \ dV=tdA$$

where t is the thickness along the z direction. The body force f has the units force/unit volume, while the traction force T has the units force/unit area. The strain-displacement relations are given by

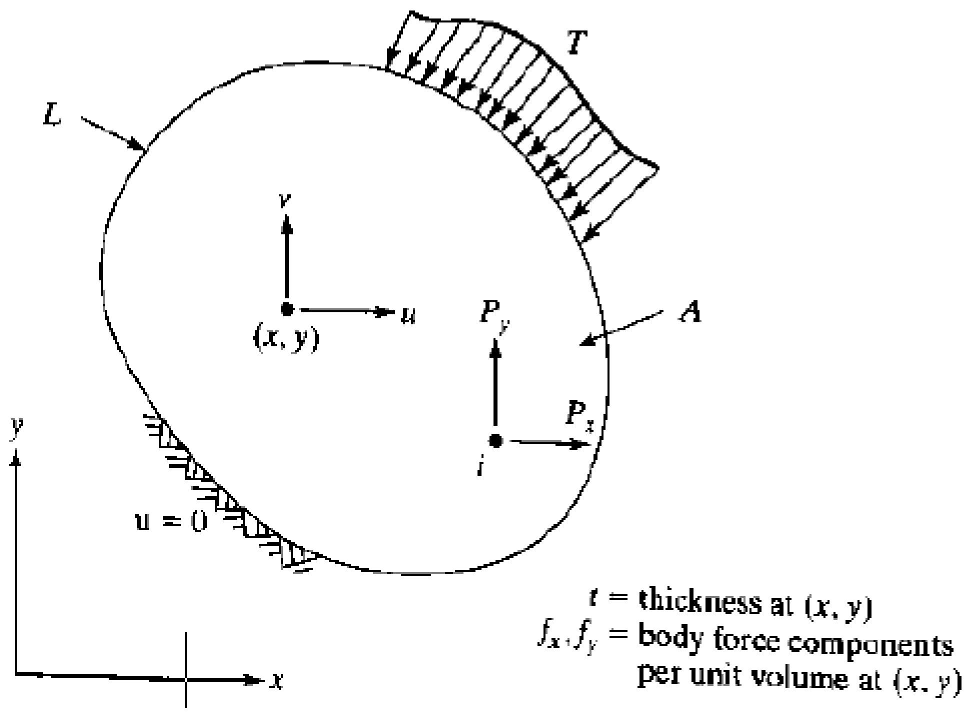


FIGURE 5.1 Two-dimensional problem.

$$\varepsilon=\left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]^T$$

Stresses and strains are related by

$\sigma=D\varepsilon$

The region is discretized with the idea of expressing the displacements in terms of values at discrete points.

Finite element modelling

The two dimensional region is divided into straight-sided triangles. Figure 5.2 shows a typical triangulation. The points where the comers of the triangles meet are ca1led *nodes,* and each triangle formed by three nodes and three sides is called an *element.* The elements fill the entire region except a small region at the boundary.1his unfilled region exists for curved boundaries, and it can be reduced by choosing smaller elements or elements with curved boundaries.1be idea of the finite element method is to solve the continuous problem approximately, and this unfilled region contributes to some part of this approximation. For the triangulation shown in Fig. 5.2, the node numbers are indicated at the corners and element numbers are circled.

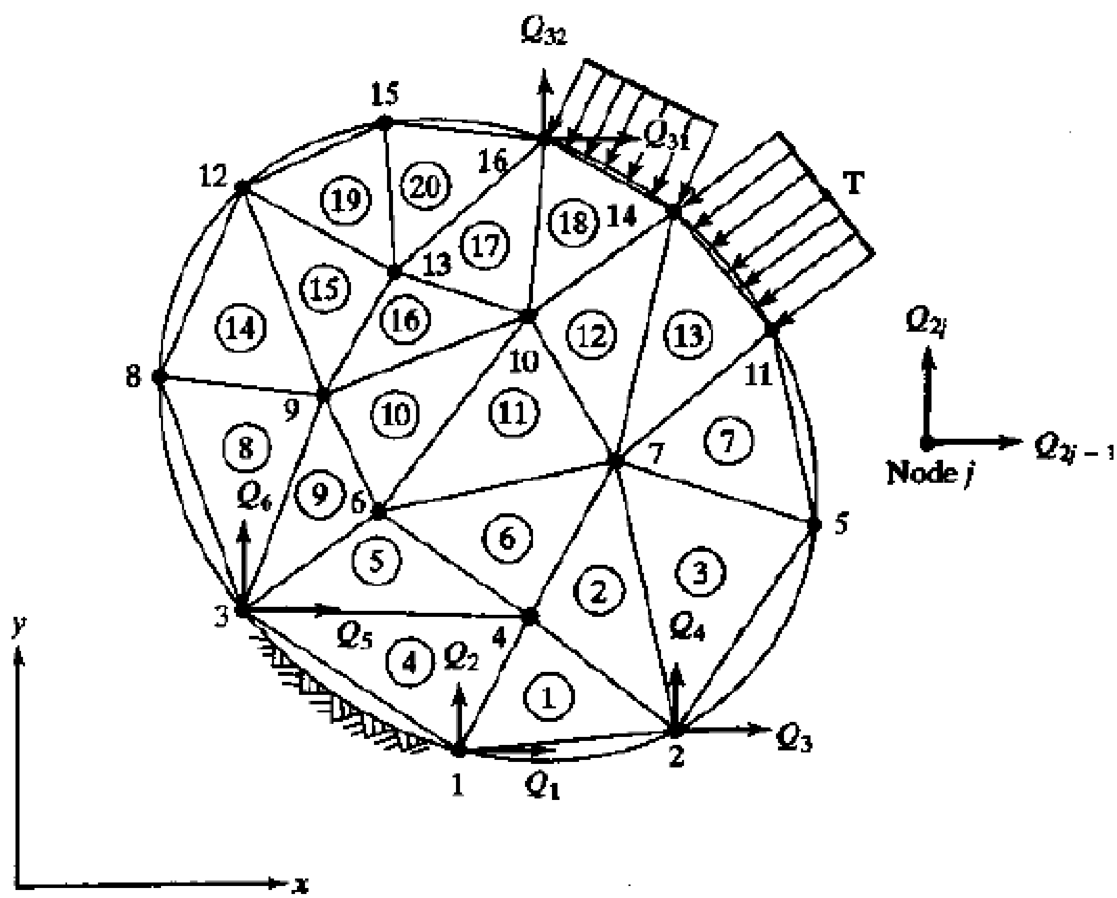


FIGURE 5.2 Finite element discretization.

In the two-dimensional problem discussed here, each node is permitted to displace in the two directions *x* and y. Thus, each node has two degrees of freedom (dofs).As seen from the numbering scheme used in trusses, the displacement components of node *j* are taken as Q_{2j-1} in the *x* direction and Q_{2j} in the *y* direction. We denote the global displacement vector as

$$\mathbf{Q}=[Q_1,Q_2,\ldots,Q_N]^T$$

where *N* is the number of degrees of freedom.

Computationally, the information on the triangulation is to be represented in the form of *nodal coordinates* and *connectivity.* The nodal coordinates are stored in a two dimensional array represented by the total number of nodes and the two coordinates per node. The connectivity may be clearly seen by isolating a typical element, as shown in Fig. 5.3. For the three nodes designated locally as 1,2, and 3, the corresponding global node numbers are defined in Fig. 5.2. This element connectivity information becomes an array of the size and number of elements and three nodes per element. A typical connectivity representation is shown in Table 5.1. Most standard finite element codes use the convention of going around the element in a counter clockwise direction to avoid calculating a negative area.

Table 5.1 establishes the correspondence of local and global node numbers and the corresponding degrees of freedom. The displacement components of a local node j in Fig. 5.3 are represented as q_{2j-l} and q_{2j} in the x and y directions, respectively. We denote the element displacement vector as

$$\boldsymbol{q}=[q_1,q_2,\ldots,q_6]^T$$

TABLE 5.1 Element Connectivity

Element number	Three nodes		
e	1	2	3
1	1	2	4
2	4	2	7
⋮			
11	6	7	10
⋮			
20	13	16	15

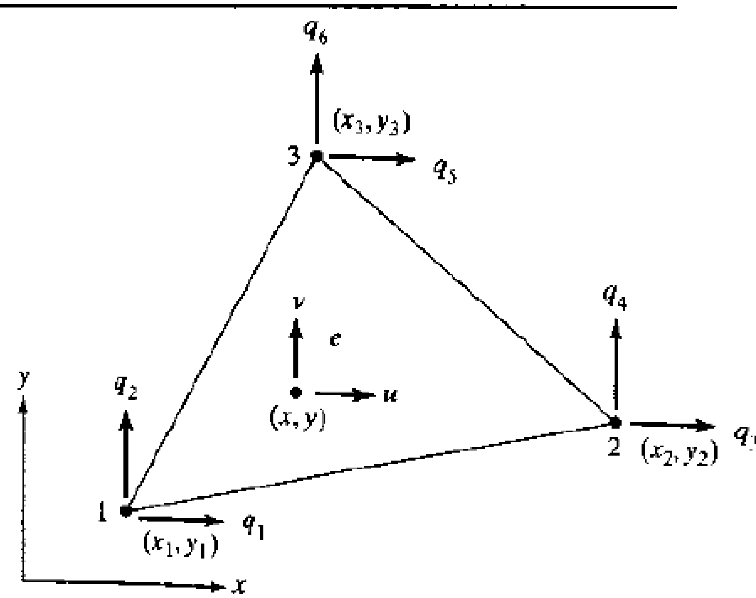


FIGURE 5.3 Triangular element.

Note that from the connectivity matrix in Table 5.1, we can extract the q vector from the global Q vector, an operation performed frequently in a finite element program. Also, the nodal coordinates designated by (x_1,y_1), (x_2,y_2), and(x_3,y_3) have the global correspondence established through Table 5.1. The local representation of nodal coordinates and degrees of freedom provides a setting for a simple and clear representation of element characteristics.

5. Applications/ Simulation/ related Laboratory example

The finite element modeling of two dimensional problems as applicable for sheet metal problems

6. MCQ-Post test

1. On gathering stiffness and loads, the system of equations is given by
   (a) KQ=F
   (b) KQ≠F
   (c) K=QF
   (d) K≠QF
2. To solve the FEM problem, it subdivides a large problem into smaller, simpler parts that are called
   (a) finite elements
   (b) infinite elements
   (c) dynamic elements
   (d) static elements
3. The applications of finite element method in two dimensional analyses are:
   (a) gravity of dams
   (b) axi-symmetric shells
   (c) stretching of plates
   (d) all

7. Conclusion

The two dimensional domain of finite element modeling effectively interpreted.

8. References

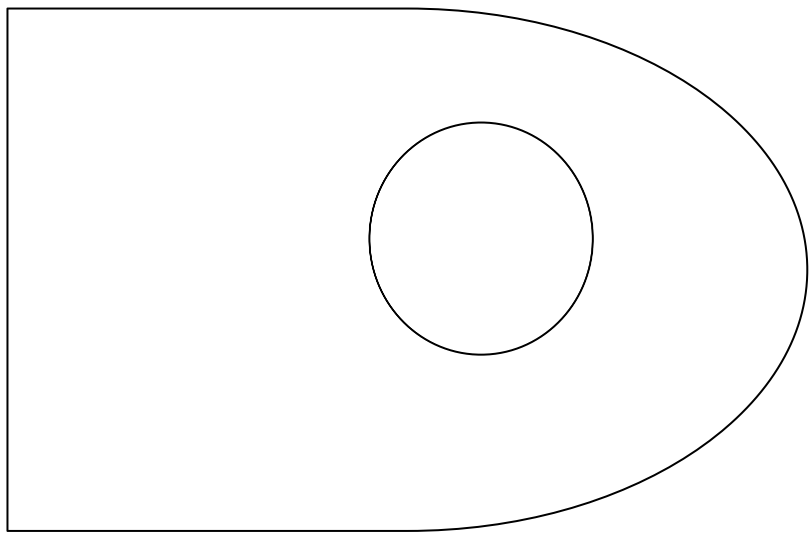
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=z8gGx0MQbzQ

10. Assignments

1. Discretize the Mechanical bracket using plane element.



UNIT-3

Name of the Course	:	**FINITE ELEMENT ANALYSIS (FEA)**
Name of the Unit	:	**Two Dimensional Continuum**
Name of the Topic	:	**Scalar valued problem – Poisson equation – Laplace equation**

1. Aim and Objectives

- To provide knowledge in 2D elements
- To study heat conduction problems using finite element method

2. Pre-Test-MCQ type

1. The quantity which has the only magnitude is called ____________

   a) A scalar quantity
   b) A vector quantity
   c) A chemical quantity
   d) A magnitude quantity

2. The equation is said to be Laplace equation if the Laplacian of a scalar filed results into _____ .

   (a) zero
   (b) positive value
   (c) negative value
   (d) infinity

3. A Laplace Transform exists when ______

   A. The function is piece-wise continuous
   B. The function is of exponential order
   C. The function is piecewise discrete
   D. The function is of differential order

   (a) **A & B**
   (b) C & D
   (c) A & D
   (d) B & C

3. Prerequisites

The knowledge of Laplace transform and basics of poission equation is required.

4. Theory behind

We consider second-order partial differential equations which involve the scalar-valued dependent variable $u = u(x, y)$. A simple example of the equations of this type is the Poisson's equation. We will present some examples of solving Poisson's and Laplace equations using the linear triangular and bilinear rectangular element.

6.1. Single Dependent Variable Problems

The finite element analysis of two-dimensional boundary value problems involves the following steps:

1. The boundary value problem is defined in a given domain Ω by a second-order partial differential equation that is subject to prescribed boundary and initial values, and
2. The boundary $\partial\Omega$ of the domain Ω is a closed curve in most problems.

Thus, the finite elements for the domain Ω are two-dimensional figures, such as triangles, rectangles, or quadrilaterals. A mesh of these elements covers the given domain, and the solution of the boundary value problem is approximated over this finite element mesh. Obviously, such a solution contains the discretization as well as approximation errors; the former error is because of the approximation of the domain, and the latter because of the approximation of the numerical solution.

We consider the general second-order equation

$$-\frac{\partial G_1}{\partial x} - \frac{\partial G_2}{\partial y} + c\,u - f = 0, \quad \text{in } \Omega, \tag{6.1}$$

where c and f are known functions of x and y, subject to the prescribed boundary conditions: $u = \hat{u}$ on Γ_1, and $-(G_1\,n_x + G_2\,n_y) = q_n$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and

$$G_1 \equiv a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}, \quad G_2 \equiv a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}, \tag{6.2}$$

with a_{ij} $(i, j = 1, 2)$, are known functions of x and y. Note that if $a_{11} = a = a_{22}$, $a_{12} = 0 = a_{21}$, and $c = 0$, then Eq (6.1) reduces to the Poisson's equation

$$-\frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a\,\frac{\partial u}{\partial y}\right) = f \quad \text{in } \Omega. \tag{6.3}$$

A mesh of quadrilateral elements in the region Ω is shown in Fig. 6.1. This mesh consists of different geometric figures of triangular, rectangular, or quadrilateral shapes. A typical element is denoted by $\Omega^{(e)}$, and the discretization error is represented as the portions of the region (shaded in Fig. 6.1) between its boundary $\Gamma \equiv \partial\Omega$ and the boundaries of the elements that lie toward the boundary Γ.

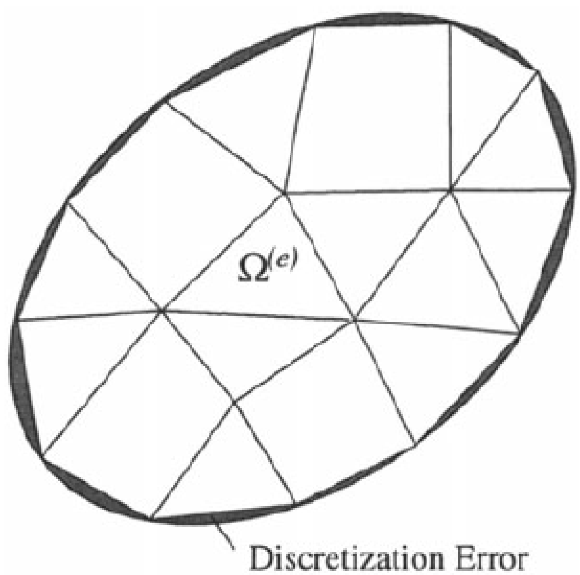


Fig. 6.1. Finite Elements.

6.1.3. Evaluation of Stiffness Matrix and Load Vector. We assume that the coefficients a_{ij} and c and the function f are constant. Then the matrix $\mathbf{K}$ and the vector $\mathbf{f}$ are evaluated as described below.

FOR A TRIANGULAR ELEMENT $\Omega^{(e)}$, the matrix $\mathbf{K}^{(e)}$ is composed of four double integrals

$$H_{ij}^{11} = \iint_{\Omega^{(e)}} \frac{\partial\phi_i^{(e)}}{\partial x}\frac{\partial\phi_j^{(e)}}{\partial x}\,dx\,dy, \qquad H_{ij}^{12} = \iint_{\Omega^{(e)}} \frac{\partial\phi_i^{(e)}}{\partial x}\frac{\partial\phi_j^{(e)}}{\partial y}\,dx\,dy,$$
$$H_{ij}^{22} = \iint_{\Omega^{(e)}} \frac{\partial\phi_i^{(e)}}{\partial y}\frac{\partial\phi_j^{(e)}}{\partial y}\,dx\,dy, \qquad H_{ij} = \iint_{\Omega^{(e)}} \phi_i^{(e)}\,\phi_j^{(e)}\,dx\,dy. \tag{6.9}$$

Thus,

$$\mathbf{K}^{(e)} = a_{11}\,\mathbf{H}^{11} + a_{12}\,\mathbf{H}^{12} + a_{21}\left(\mathbf{H}^{12}\right)^T + a_{22}\,\mathbf{H}^{22} + c\,\mathbf{H}. \tag{6.10}$$

The vector $\mathbf{f}^{(e)}$ is defined by

$$f_j^{(e)} = \iint_{\Omega^{(e)}} f\,\phi_j\,dx\,dy. \tag{6.11}$$

Note that the integrals in (6.10) and (6.11) are of the type

$$I_{mn} = \iint_{\Omega^{(e)}} x^m y^n \, dx \, dy. \tag{6.12}$$

Using formula (5.18) the integrals I_{mn} for $m, n = 0, 1, 2,$ have the following values:

$$\begin{aligned}
&I_{00} = A^{(e)} \equiv |\Omega^{(e)}|, \quad I_{10} = A^{(e)}\,\hat{x}, \quad \hat{x} = \frac{1}{3}\sum_{k=1}^{3} x_k, \\
&I_{01} = A^{(e)}\,\hat{y}, \quad \hat{y} = \frac{1}{3}\sum_{k=1}^{3} y_k, \quad I_{11} = \frac{A^{(e)}}{12}\left(\sum_{k=1}^{3} x_k y_k + 9\,\hat{x}\hat{y}\right), \\
&I_{20} = \frac{A^{(e)}}{12}\left(\sum_{k=1}^{3} x_k^2 + 9\,\hat{x}^2\right), \quad I_{02} = \frac{A^{(e)}}{12}\left(\sum_{k=1}^{3} y_k^2 + 9\,\hat{y}^2\right).
\end{aligned} \tag{6.13}$$

Then, using the results in (5.3)–(5.4) we have $\dfrac{\partial \phi_i^{(e)}}{\partial x} = b_i^{(e)}$, and $\dfrac{\partial \phi_i^{(e)}}{\partial y} = c_i^{(e)}$, which, in view of formulas (6.9), yield

$$\begin{aligned}
H_{ij}^{11} &= A^{(e)}\, b_i^{(e)} b_j^{(e)}, \\
H_{ij}^{12} &= A^{(e)}\, b_i^{(e)} c_j^{(e)}, \\
H_{ij}^{22} &= A^{(e)}\, c_i^{(e)} c_j^{(e)}, \\
H_{ij} &= A^{(e)}\left[a_i^{(e)} a_j^{(e)} + \left(a_i^{(e)} b_j^{(e)} + a_j^{(e)} b_i^{(e)}\right)\hat{x} + \left(a_i^{(e)} c_j^{(e)} + c_i^{(e)} a_j^{(e)}\right)\hat{y}\right] \\
&\qquad + b_i^{(e)} b_j^{(e)}\, I_{20} + \left(b_i^{(e)} c_j^{(e)} + b_j^{(e)} c_i^{(e)}\right) I_{11} + c_i^{(e)} c_j^{(e)}\, I_{02}, \\
f_j^{(e)} &= \frac{f^{(e)} A^{(e)}}{3}, \quad Q_j^{(e)} = \frac{q_n A^{(e)}}{3}.
\end{aligned} \tag{6.14}$$

The values of $\mathbf{K}^{(e)}$ and $\mathbf{f}^{(e)}$ are then evaluated for each element $\Omega^{(e)}$ from the data (coordinates) of the nodes.

5. Applications/ Simulation/ related Laboratory example

The application of triangular elements is constructively useful for 2D problems

6. MCQ-Post test

1. In free space, the Poisson equation becomes
   (a) Maxwell equation
   (b) Ampere equation
   (c) Laplace equation
   (d) Steady state equation

2. Suppose the potential function is a step function. The equation that gets satisfied is
   (a) Laplace equation
   (b) Poisson equation
   (c) Maxwell equation
   (d) Ampere equation
3. Poisson equation can be derived from which of the following equations?
   (a) Point form of Gauss law
   (b) Integral form of Gauss law
   (c) Point form of Ampere law
   (d) Integral form of Ampere law

4. Poisson's and Laplace equations can be easily derived from ____________.
   (a) Coulomb's law
   (b) Gauss law
   (c) Ampere's law
   (d) Faraday's law

7. Conclusion

The implementation of Poisson and Laplace equations were discussed with 2D domain.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=nnc6HBvyxeI

10. Assignments

1. Write short notes on effectiveness of Poisson and Laplace equations for two dimensional problems.

UNIT-3

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Two Dimensional Continuum**

Name of the Topic : **Triangular elements – Element stiffness matrix – Force vector**

1. Aim and Objectives

- To understand the use of 2D Triangular elements
- To familiarize the Element stiffness matrix and Force vector for 2D domain.

2. Pre-Test-MCQ type

1. As the number of elements is increased, the problem converges to
   (a) Exact solution
   (b) Partial exact solution
   (c) Approximate solution
   (d) All of these
2. Convergence is a process of
   a. Dividing the domain
   b. Converting local coordinates into natural coordinates
   c. Arriving at a solution that is close to the exact solution
   d. Arriving at a solution that is far from the exact solution

3. Prerequisites

The technical information of engineering mathematics is needed.

4. Theory behind

The displacements at points inside an element need to be represented in terms of the nodal displacements of the element. As discussed earlier, the finite element method uses the concept of shape functions in systematically developing these interpolations. For the constant strain triangle. the shape functions are linear over the element. The three shape functions N_1. N_2, and N_3 corresponding to nodes1, 2, and 3, respectively. are shown in Fig. 5.4. Shape function *Nt* is 1 at node 1 and linearly reduces to 0 at nodes 2 and 3. The values of shape function N_1 thus define a plane surface shown shaded in Fig. 5.4a. N_2 and N_3 are represented by similar surfaces having values of 1 at nodes 2 and 3, respectively, and dropping to 0 at the opposite edges. Any linear combination of these shape functions also represents a plane surface. In particular, $N_1 + N_2 + N_3$ represents a plane at a height of 1 at nodes 1,2, and 3, and, thus, it is parallel to the triangle 123. Consequently, for every N_1. N_2, and N_3.

$N_1 + N_2 + N_3 = 1$

N_1, N_2, and N_3 are therefore not linearly independent; only two of these are independent. The independent shape functions are conveniently represented by the pair ξ, η as

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta \tag{5.10}$$

where ξ, η are natural coordinates (Fig. 5.4). At this stage, the similarity with the one-dimensional element (Chapter 3) should be noted: in the one-dimensional problem the x-coordinates were mapped onto the ξ coordinates, and shape functions were defined as functions of ξ. Here, in the two-dimensional problem, the x-, y-coordinates are mapped onto the ξ-, η-coordinates, and shape functions are defined as functions of ξ and η.

The shape functions can be physically represented by **area coordinates**. A point (x, y) in a triangle divides it into three areas, A_1, A_2, and A_3, as shown in Fig. 5.5. The shape functions N_1, N_2, and N_3 are precisely represented by

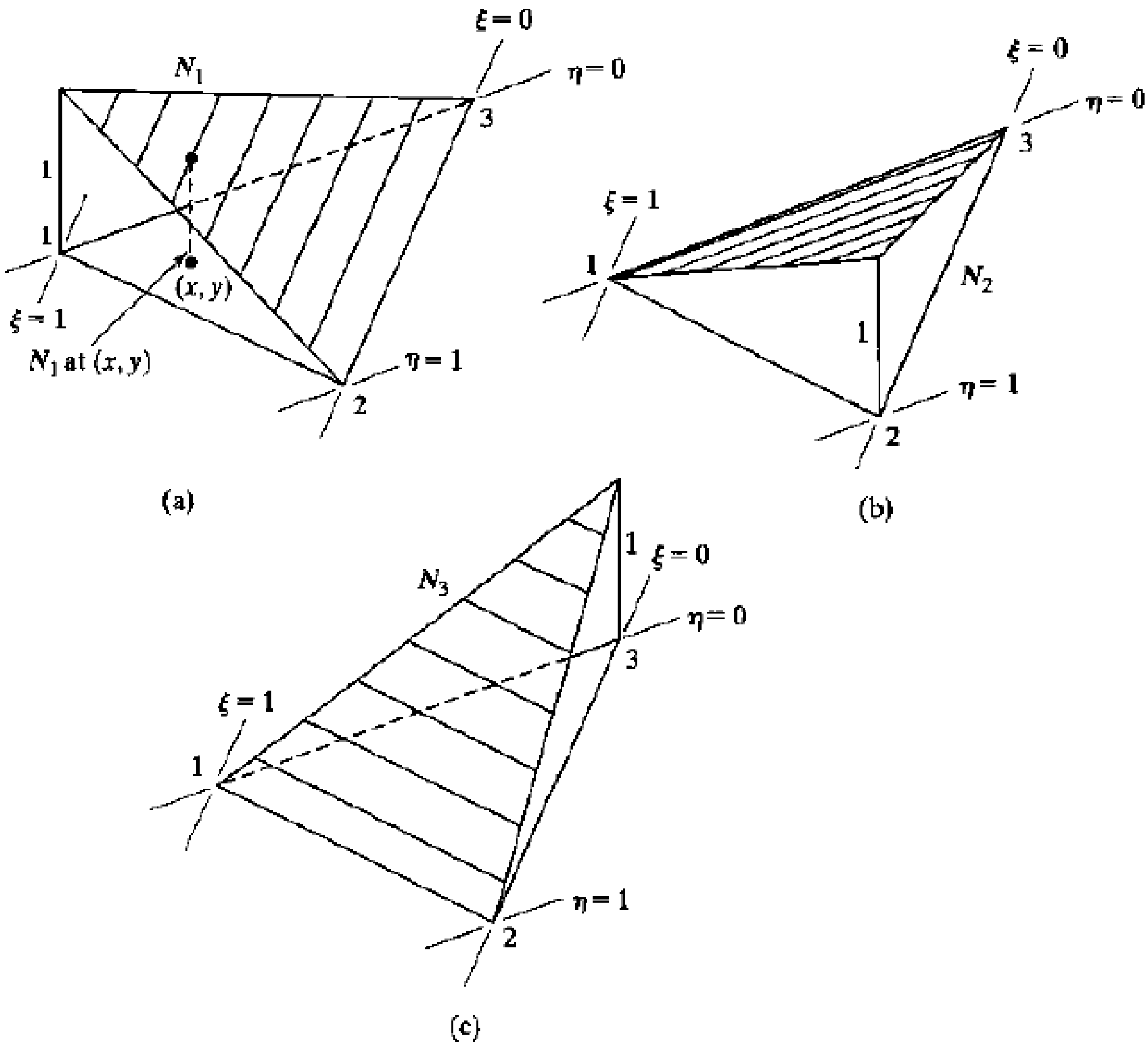


FIGURE 5.4 Shape functions.

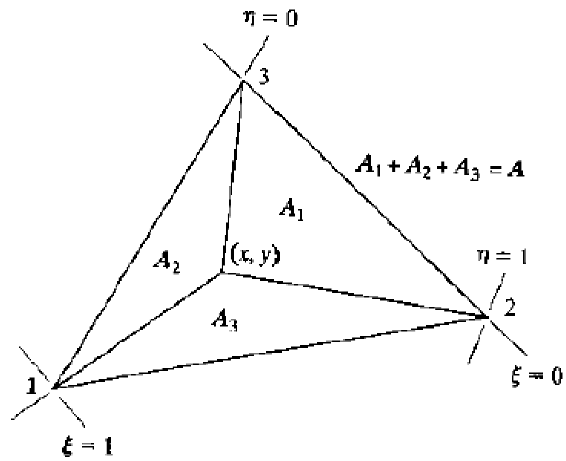


FIGURE 5.5 Area coordinates.

$$N_1 = \frac{A_1}{A} \qquad N_2 = \frac{A_2}{A} \qquad N_3 = \frac{A_3}{A} \tag{5.11}$$

where A is the area of the element. Clearly, $N_1 + N_2 + N_3 = 1$ at every point inside the triangle.

Element Stiffness

We now substitute for the strain from the element strain–displacement relationship in Eq. 5.25 into the element strain energy U_e in Eq. 5.28b, to obtain

$$\begin{aligned} U_e &= \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA \\ &= \frac{1}{2}\int_e \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}t\,dA \end{aligned} \tag{5.29a}$$

Taking the element thickness t_e as constant over the element and remembering that all terms in the **D** and **B** matrices are constants, we have

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t_e\left(\int_e dA\right)\mathbf{q} \tag{5.29b}$$

Now, $\int_e dA = A_e$, where A_e is the area of the element. Thus,

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q} \tag{5.29c}$$

or

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{5.29d}$$

where $\mathbf{k}^e$ is the element stiffness matrix given by

$$\mathbf{k}^e = t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \tag{5.30}$$

Force Terms

The body force term $\int_e \mathbf{u}^T\mathbf{f}t\,dA$ appearing in the total potential energy in Eq. 5.28b is considered first. We have

$$\int_e \mathbf{u}^T\mathbf{f}t\,dA = t_e \int_e (uf_x + vf_y)\,dA$$

Using the interpolation relations given in Eq. 5.12a, we find that

$$\begin{aligned}\int_e \mathbf{u}^T\mathbf{f}t\,dA = {} & q_1\left(t_e f_x \int_e N_1 dA\right) + q_2\left(t_e f_y \int_e N_1\,dA\right) \\ & + q_3\left(t_e f_x \int_e N_2\,dA\right) + q_4\left(t_e f_y \int_e N_2\,dA\right) \\ & + q_5\left(t_e f_x \int_e N_3\,dA\right) + q_6\left(t_e f_y \int_e N_3\,dA\right)\end{aligned} \tag{5.33}$$

From the definition of shape functions on a triangle, shown in Fig. 5.4, $\int_e N_1\,dA$ represents the volume of a tetrahedron with base area A_e and height of corner equal to 1 (nondimensional). The volume of this tetrahedron is given by $\frac{1}{3}$ × Base area × Height (Fig. 5.6) as in

$$\int_e N_i\,dA = \tfrac{1}{3}A_e \tag{5.34}$$

Similarly, $\int_e N_2\,dA = \int_e N_3\,dA = \frac{1}{3}A_e$, Equation 5.33 can now be written in the form

$$\int_e \mathbf{u}^T\mathbf{f}t\,dA = \mathbf{q}^T\mathbf{f}^e \tag{5.35}$$

5. Applications/ Simulation/ related Laboratory example

The application of constant strain triangular element is widely used in various plane stress and strain problems.

6. MCQ-Post test

1. When thin plate is subjected to loading in its own plane only, the condition is called

    (a) Plane Stress

    (b) Plane strain

    (c) Zero stress

    (d) Zero strain

2. Identify the sequence of steps in Finite Element Method:

1. Solving for primary variables

2. Imposition of boundary conditions

3. Post processing

4. Finite Element Discretization

5. Assemblage.

6. Deriving element equations

(a) 1-2-3-4-5-6
(b) 2-1-4-3-6-5
(c) 4-1-5-2-6-3
(d) 4-6-5-2-1-3

3. A 2-D structural element is a
   (a) Truss Element
   (b) Beam element
   (c) CST element
   (d) All of them

4. Number of displacements for 3-noded CST element is
   (a) 7
   (b) 6
   (c) 4
   (d) 5
5. The determinant of an CST element stiffness matrix is always
   (a) One
   (b) zero
   (c) depends on size of [K]
   (d) Two

6. Example for plane stress problem is
   (a) Strip footing resting on soil mass
   (b) A thin plate loaded in a plane
   (c) A long cylinder
   (d) A gravity dam

7. Conclusion

The 3 noded CST element and associated shape functions were interpreted.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON “Fundamentals of Finite Element Analysis”2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
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- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

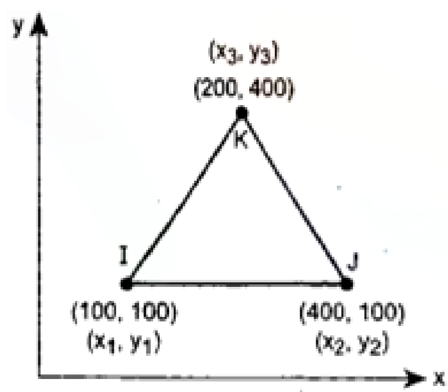
9. Video

https://www.youtube.com/watch?v=m6u4lOK6RyY

https://www.youtube.com/watch?v=DCGm0qgIXcs

10. Assignments

1. For the CST element shown in Figure, find the element stiffness matrix. Take t= 20mm and E= 2×10^5 N/mm^2.



UNIT-3

Name of the Course	:	**FINITE ELEMENT ANALYSIS (FEA)**
Name of the Unit	:	**Two Dimensional Continuum**
Name of the Topic	:	**Galerkin approach - Stress calculation – Temperature effects**

1. Aim and Objectives

- To learn the formulation of stiffness matrix for CST element using galerkin approach
- To find the stress and temperature for 2D problems.

2. Pre-Test-MCQ type

1. The elastic stress strain behaviour of rubber is
   (a) Linear
   (b) Non-linear
   (c) Plastic
   (d) No fixed relationship
2. Effect of a force on a body depends upon
   (a) Magnitude
   (b) Direction
   (c) Position or line of action
   (d) All of the above

3. The value of Poisson's ratio for steel is between

   (a)0.01 to 0.1
   (b)0.23 to 0.27
   (c)0.25 to 0.33
   (d)0.4 to 0.6

3. Prerequisites

1. The knowledge of engineering mechanics, strength of materials, Materials science and engineering are required.
2. The fundamental knowledge of basics of engineering mathematics

4. Theory behind

Galerkin Approach

Following the steps presented in Chapter 1, we introduce

$$\boldsymbol{\phi} = [\phi_x, \phi_y]^{T} \tag{5.46}$$

and

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \left[\frac{\partial \phi_x}{\partial x}, \frac{\partial \phi_y}{\partial y}, \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right]^{T} \tag{5.47}$$

where ϕ is an arbitrary (virtual) displacement vector, consistent with the boundary conditions. The variational form is given by

$$\int_A \boldsymbol{\sigma}^{T}\boldsymbol{\epsilon}(\boldsymbol{\phi})t\,dA - \left(\int_A \boldsymbol{\phi}^{T}\mathbf{f}t\,dA + \int_L \boldsymbol{\phi}^{T}\mathbf{T}t\,d\ell + \sum_i \boldsymbol{\phi}_i^{T}\mathbf{P}_i\right) = 0 \tag{5.48}$$

where the first term represents the internal virtual work. The expression in parentheses represents the external virtual work. On the discretized region, the previous equation becomes

$$\sum_e \int_e \boldsymbol{\epsilon}^{T}\mathbf{D}\boldsymbol{\epsilon}(\boldsymbol{\phi})t\,dA - \left(\sum_e \int_e \boldsymbol{\phi}^{T}\mathbf{f}t\,dA + \int_L \boldsymbol{\phi}^{T}\mathbf{T}t\,d\ell + \sum_i \boldsymbol{\phi}_i^{T}\mathbf{P}_i\right) = 0 \tag{5.49}$$

Using the interpolation steps of Eqs. 5.12–5.14, we express

$$\boldsymbol{\phi} = \mathbf{N}\boldsymbol{\psi} \tag{5.50}$$

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \mathbf{B}\boldsymbol{\psi} \tag{5.51}$$

where

$$\boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6]^{T} \tag{5.52}$$

represents the arbitrary nodal displacements of element e. The global nodal displacement variations $\boldsymbol{\Psi}$ are represented by

$$\boldsymbol{\Psi} = [\Psi_1, \Psi_2, \ldots, \Psi_N]^{T} \tag{5.53}$$

The element internal work term in Eq. 5.49 can be expressed as

$$\int_e \boldsymbol{\epsilon}^{T}\mathbf{D}\boldsymbol{\epsilon}(\boldsymbol{\phi})t\,dA = \int_e \mathbf{q}^{T}\mathbf{B}^{T}\mathbf{D}\mathbf{B}\boldsymbol{\psi}t\,dA$$

Noting that all terms of **B** and **D** are constant and denoting t_e and A_e as thickness and area of element, respectively, we find that

$$\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}(\boldsymbol{\phi})t\,dA = \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t_e\int_e dA\,\boldsymbol{\psi}$$
$$= \mathbf{q}^{\mathrm{T}}t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\boldsymbol{\psi} \tag{5.54}$$
$$= \mathbf{q}^{\mathrm{T}}\mathbf{k}^e\boldsymbol{\psi}$$

where $\mathbf{k}^e$ is the element stiffness matrix given by

$$\mathbf{k}^e = t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \tag{5.55}$$

Stress Calculations

Since strains are constant in a constant-strain triangle (CST) element, the corresponding stresses are constant. The stress values need to be calculated for each element. Using the stress–strain relations in Eq. 5.6 and element strain–displacement relations in Eq. 5.25, we have

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{q} \tag{5.60}$$

The connectivity in Table 5.1 is once again needed to extract the element nodal displacements $\mathbf{q}$ from the global displacements vector $\mathbf{Q}$. Equation 5.60 is used to calculate the element stresses. For interpolation purposes, the calculated stress may be used as the value at the centroid of the element.

Principal stresses and their directions are calculated using Mohr's circle relationships. The program at the end of the chapter includes the principal stress calculations.

Detailed calculations in Example 5.6 illustrate the steps involved. However, it is expected that the exercise problems at the end of the chapter will be solved using a computer.

Temperature Effects

If the distribution of the change in temperature $\Delta T(x, y)$ is known, the strain due to this change in temperature can be treated as an initial strain $\boldsymbol{\epsilon}_0$. From the theory of mechanics of solids, $\boldsymbol{\epsilon}_0$ can be represented by

$$\boldsymbol{\epsilon}_0 = [\alpha\Delta T, \alpha\Delta T, 0]^{\mathrm{T}} \tag{5.61}$$

for plane stress and

$$\boldsymbol{\epsilon}_0 = (1 + \nu)[\alpha\Delta T, \alpha\Delta T, 0]^{\mathrm{T}} \tag{5.62}$$

for plane strain. The stresses and strains are related by

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \tag{5.63}$$

The effect of temperature can be accounted for by considering the strain energy term. We have

The effect of temperature can be accounted for by considering the strain energy term. We have

$$U = \frac{1}{2}\int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)^{\mathrm{T}}\mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)t\, dA$$

$$= \frac{1}{2}\int (\boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon} - 2\boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}_0^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0)t\, dA \qquad (5.64)$$

The first term in the previous expansion gives the stiffness matrix derived earlier. The last term is a constant, which has no effect on the minimization process. The middle term, which yields the temperature load, is now considered in detail. Using the strain-displacement relationship $\boldsymbol{\epsilon} = \mathbf{Bq}$,

$$\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 t\, dA = \sum_e \mathbf{q}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0)t_e A_e \qquad (5.65)$$

This step is directly obtained in the Galerkin approach where $\boldsymbol{\epsilon}^{\mathrm{T}}$ will be $\boldsymbol{\epsilon}^{\mathrm{T}}(\boldsymbol{\phi})$ and $\mathbf{q}^{\mathrm{T}}$ will be $\boldsymbol{\psi}^{\mathrm{T}}$.

It is convenient to designate the element temperature load as

$$\boldsymbol{\Theta}^e = t_e A_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}_0 \qquad (5.66)$$

where

$$\boldsymbol{\Theta}^e = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6]^{\mathrm{T}} \qquad (5.67)$$

The vector $\boldsymbol{\epsilon}_0$ is the strain in Eq. 5.61 or 5.62 due to the average temperature change in the element. $\boldsymbol{\Theta}^e$ represents the element nodal load contributions that must be added to the global force vector using the connectivity.

The stresses in an element are then obtained by using Eq. 5.63 in the form

$$\boldsymbol{\sigma} = \mathbf{D}(\mathbf{Bq} - \boldsymbol{\epsilon}_0) \qquad (5.68)$$

5. Applications/ Simulation/ related Laboratory example

The 2D FEA simulations are widely applicable for plane stress and strain problems(i.e. sheet metals processing)

6. MCQ-Post test

1. When a thin plate is subjected to loading in its own plane only, the condition is called _________.
   (a) plane stress
   (b) plane strain
   (c) zero stress
   (d) zero strain

2. In CST element _________ is constant
   (a) Stress
   (b) Strain
   (c) shape function
   (d) All

3. Stiffness matrix for 2D CST element
   (a) B^TDBA-t
   (b) B^TD- BAt
   (c) B^TDB+At
   (d) B^TDBAt

4. A three noded triangular element has a stiffness matrix of order

   (a) 2 x 2
   (b) 4 x 4
   (c) **6 x 6**
   (d) 1 x 1

5. Stress – Strain relationship matrix for two dimensional plane stress

   (a) 5x 5(b) 4 x 4 (c) 6 x 6 (d) **3 x 3**

6. In 2D finite element analysis, when thickness is very small as compared to the size of the domain, which of the following condition should be considered?

   a. Serendipity conditions
   b. Plane strain conditions
   c. Axis-symmetric conditions
   d. **Plane stress conditions**

7. A 2D CST strain-displacement matrix[B] of order
   a) 2 x 2
   b) 4 x 4
   c) 3x 6
   d) 3 x 4

8. A 2D CST strain-displacement matrix[B] of order
   a) 2 x 2
   b) 4 x 4
   c) 3x 6
   d) 3 x 4

9. Conclusion

The evaluation of Stress calculation and Temperature effects were discussed.

10.References

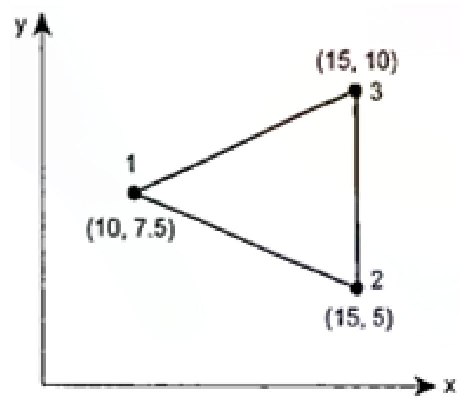
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

11.Video

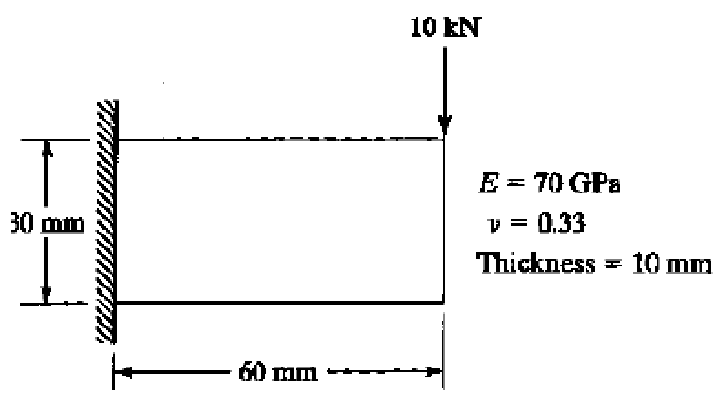
https://www.youtube.com/watch?v=yWRqyWBtnAc
https://www.youtube.com/watch?v=3WtCixoFHIY

12.Assignments

1. Find the element stresses for CST element shown in Figure.



2. Solve the plane stress problem in Figure using three different mesh divisions. Compare your deformation and stress results with values obtained from elementary beam theory.



UNIT-4

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Axisymmetric Continuum**

Name of the Topic : **Axisymmetric formulation**

1. Aim and Objectives

- To focus on the Axisymmetric formulation for a Axisymmetric Continuum

2. Pre-Test-MCQ type

1. Global axes are
   (a) **defined for the entire system.**
   (b) defined for co-ordinates
   (c) defined for both entire system and co-ordinates
   (d) none of these
2. Characteristic of shape function is
   (a) It has unit value at one nodal point and zero value at other nodal points
   (b) The sum of shape function is equal to one.
   (c) Both (a) and(b) correct
   (d) Both (a) and(b) incorrect

3. Prerequisites

The engineering mathematics and engineering skill on axisymmetric domain required.

4. Theory behind

Problems involving three-dimensional axisymmetric solids or solids of revolution, subjected to axisymmetric loading, reduce to simple two-dimensional problems. Because of total symmetry about the z-axis, as seen in Fig. 6.1a, all deformations and stresses are independent of the rotational angle θ. Thus, the problem needs to be looked at as a twodimensional problem in *rz,* defined on the revolving area (Fig. 6.1b). Gravity forces can be considered if acting in the *z* direction. Revolving bodies like flywheels can be analyzed by introducing centrifugal forces in the body force term. We now discuss the axisymmetric problem formulation.

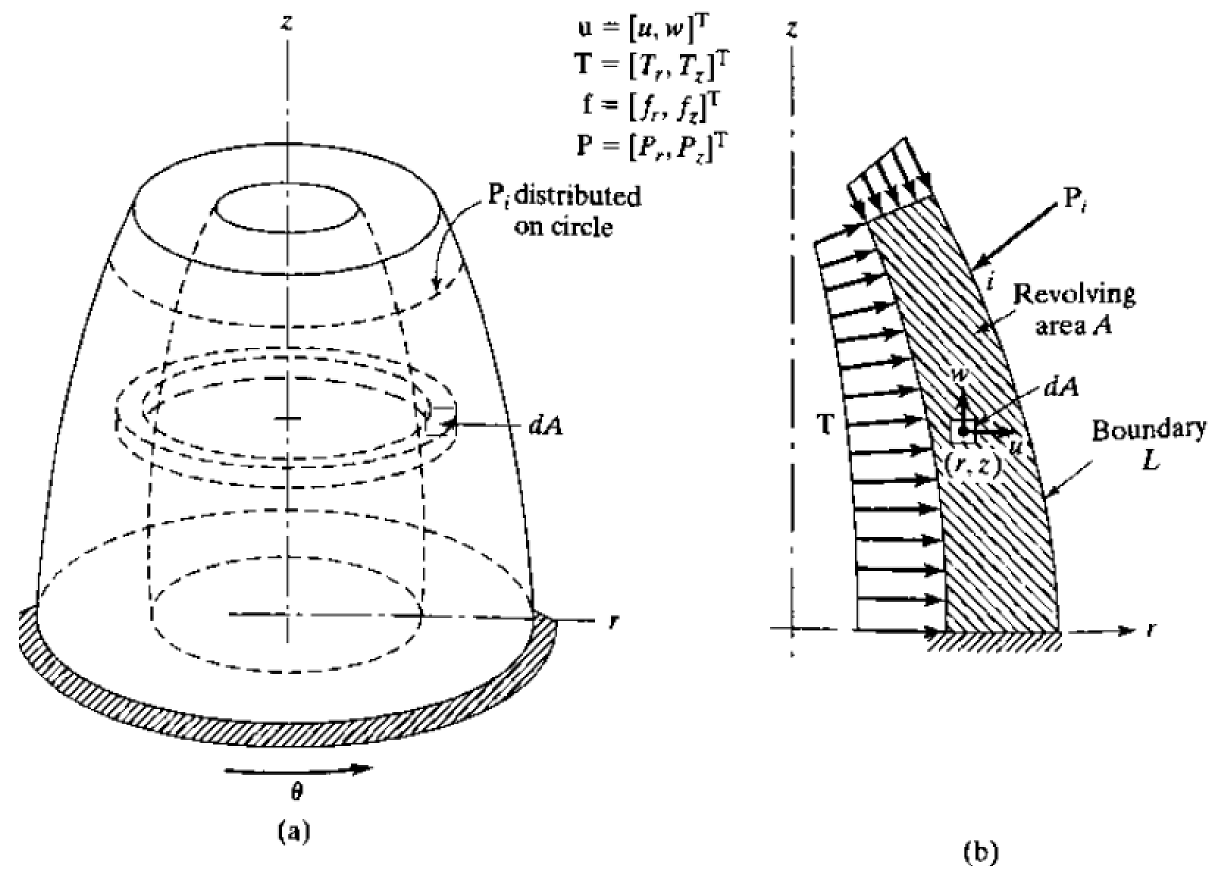


FIGURE 6.1 Axisymmetric problem.

6.2 AXISYMMETRIC FORMULATION

Considering the elemental volume shown in Fig. 6.2, the potential energy can be written in the form

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon} r\, dA\, d\theta - \int_0^{2\pi}\int_A \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA\, d\theta - \int_0^{2\pi}\int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} r\, d\ell\, d\theta - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{6.1}$$

where $r\, d\ell\, d\theta$ is the elemental surface area and the point load $\mathbf{P}_i$ represents a line load distributed around a circle, as shown in Fig. 6.1.

All variables in the integrals are independent of θ. Thus, Eq. 6.1 can be written as

$$\Pi = 2\pi\left(\frac{1}{2}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon} r\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} r\, d\ell\right) - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{6.2}$$

where

$$\mathbf{u} = [u, w]^{\mathrm{T}} \tag{6.3}$$

$$\mathbf{f} = [f_r, f_z]^{\mathrm{T}} \tag{6.4}$$

$$\mathbf{T} = [T_r, T_z]^{\mathrm{T}} \tag{6.5}$$

From Fig. 6.3, we can write the relationship between strains $\boldsymbol{\epsilon}$ and displacements $\mathbf{u}$ as

$$\boldsymbol{\epsilon} = [\epsilon_r, \epsilon_z, \gamma_{rz}, \epsilon_\theta]^{\mathrm{T}}$$
$$= \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]^{\mathrm{T}} \tag{6.6}$$

The stress vector is correspondingly defined as

$$\boldsymbol{\sigma} = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]^{\mathrm{T}} \tag{6.7}$$

The stress–strain relations are given in the usual form, viz.,

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{6.8}$$

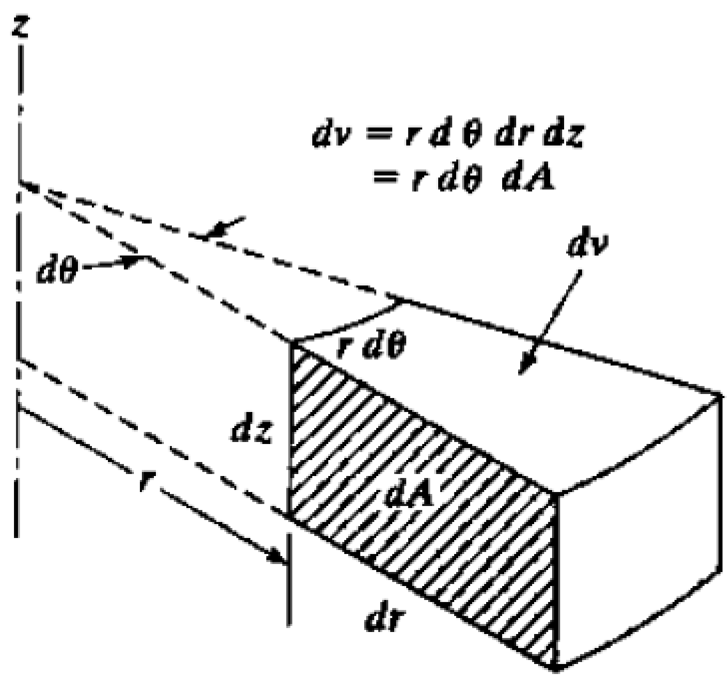


FIGURE 6.2 Elemental volume.

5. Applications/ Simulation/ related Laboratory example

The application of axisymmetric used in many engineering disciplines.

6. MCQ-Post test

1. A symmetric structure can be analyzed by modeling one symmetric part
   - (a) Depending on applied loads
   - (b) Depending on boundary conditions
   - (c) Always yes
   - **(d) Depending on applied loads & boundary conditions**

2. The example of cooling tower is considered as
    (a) Plane stress
    (b) Plane strain
    (c) Axisymmetric
    (d) none of the these
3. Open ended thin cylinder is considered as
    (a) Axisymmetric
    (b) Plane strain
    (c) Plane stress
    (d) All of the these

7. Conclusion

The axisymmetric formulation and relative forces and, boundary conditions were discussed.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=sZQ_pX_Lus4

10. Assignments

1. Write short notes about the formulation of axisymmetric condition

UNIT-4

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Axisymmetric Continuum**

Name of the Topic : **Element stiffness matrix and force vector**

1. Aim and Objectives

- To Evaluate the element stiffness matrix and force vector

2. Pre-Test-MCQ type

1. Global stiffness matrix size is calculated by
   (a) Number of nodes × Degrees of freedom per node
   (b) Number of nodes + Degrees of freedom per node
   (c) Number of nodes -Degrees of freedom per node
   (d) None of these
2. What are the conditions for a problem to be axisymmetric?
   (a) The problem domain must be symmetric about the axis of revolution All
   (b) Boundary condition must be symmetric about the axis of revolution All
   (c) Loading condition must be symmetric about the axis of revolution
   (d) All of these

3. Prerequisites

The engineering mathematics and engineering skill on axisymmetric domain required.

4. Theory behind

FINITE ELEMENT MODELING: TRIANGULAR ELEMENT

The two-dimensional region defined by the revolving area is divided into triangular elements, as shown in Fig. 6.4. Though each element is completely represented by the area in the rz plane, in reality, it is a ring-shaped solid of revolution obtained by revolving the triangle about the z-axis. A typical element is shown in Fig. 6.5.

The definition of connectivity of elements and the nodal coordinates follow the steps involved in the CST element discussed in Section 5.3. We note here that the r- and z-coordinates, respectively, replace x and y.

Using the three shape functions N_1, N_2, and N_3, we define

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{6.13}$$

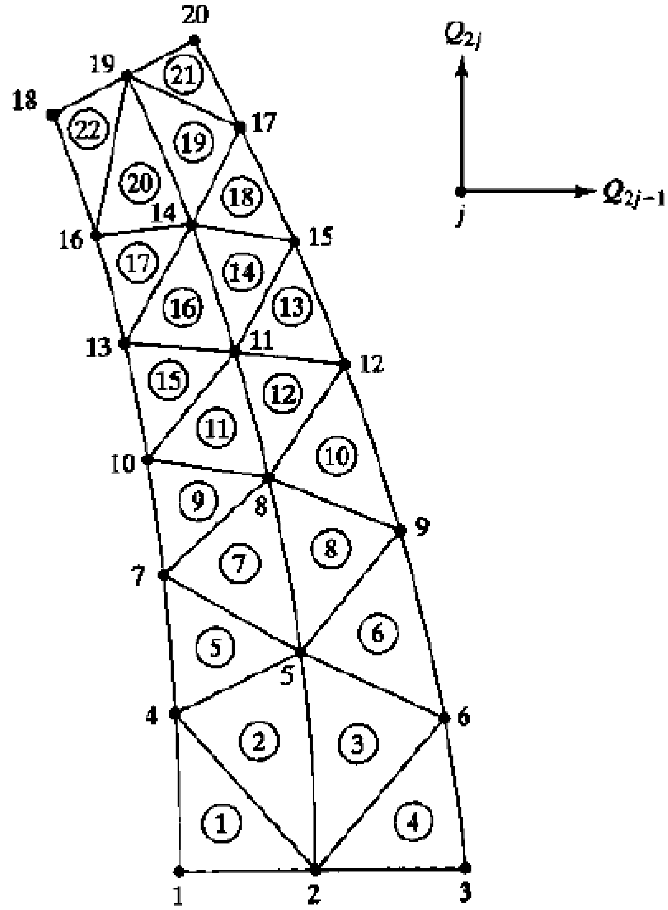
where $\mathbf{u}$ is defined in (6.3) and

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{6.14}$$

$$\mathbf{q} = [q_1, \quad q_2, \quad q_3, \quad q_4, \quad q_5, \quad q_6]^T \tag{6.15}$$

If we denote $N_1 = \xi$ and $N_2 = \eta$, and note that $N_3 = 1 - \xi - \eta$, then Eq. 6.13 gives

$$\begin{aligned} u &= \xi q_1 + \eta q_3 + (1 - \xi - \eta) q_5 \\ w &= \xi q_2 + \eta q_4 + (1 - \xi - \eta) q_6 \end{aligned} \tag{6.16}$$



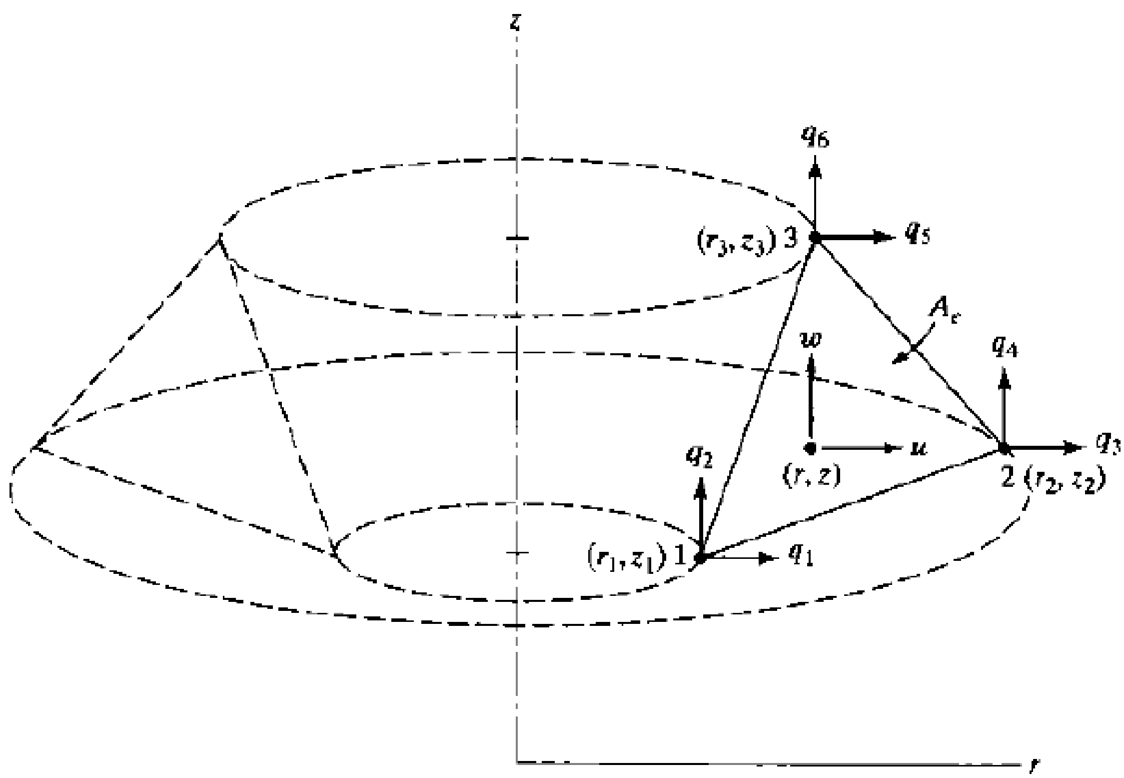


FIGURE 6.5 Axisymmetric triangular element.

By using the isoparametric representation, we find

$$\begin{aligned} r &= \xi r_1 + \eta r_2 + (1 - \xi - \eta) r_3 \\ z &= \xi z_1 + \eta z_2 + (1 - \xi - \eta) z_3 \end{aligned} \tag{6.17}$$

The chain rule of differentiation gives

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} \tag{6.18}$$

and

$$\begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} \tag{6.19}$$

where the Jacobian is given by

$$\mathbf{J} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \tag{6.20}$$

In the definition of **J** earlier, we have used the notation $r_{ij} = r_i - r_j$ and $z_{ij} = z_i - z_j$. The determinant of **J** is

$$\det \mathbf{J} = r_{13} z_{23} - r_{23} z_{13} \tag{6.21}$$

Recall that $|\det \mathbf{J}| = 2A_e$. That is, the absolute value of the determinant of **J** equals twice the area of the element. The inverse relations for Eqs. 6.18 and 6.19 are given by

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} \tag{6.22}$$

where

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & z_{13} \end{bmatrix} \tag{6.23}$$

Introducing these transformation relationships into the strain–displacement relations in Eq. 6.6 and using Eqs. 6.16, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_2 - q_6) + r_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_1 - q_5) + r_{13}(q_3 - q_5) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{Bmatrix}$$

This can be written in the matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{6.24}$$

where the element strain–displacement matrix, of dimension (4×6), is given by

$$\mathbf{B} = \begin{bmatrix} \frac{z_{23}}{\det \mathbf{J}} & 0 & \frac{z_{31}}{\det \mathbf{J}} & 0 & \frac{z_{12}}{\det \mathbf{J}} & 0 \\ 0 & \frac{r_{32}}{\det \mathbf{J}} & 0 & \frac{r_{13}}{\det \mathbf{J}} & 0 & \frac{r_{21}}{\det \mathbf{J}} \\ \frac{r_{32}}{\det \mathbf{J}} & \frac{z_{23}}{\det \mathbf{J}} & \frac{r_{13}}{\det \mathbf{J}} & \frac{z_{31}}{\det \mathbf{J}} & \frac{r_{21}}{\det \mathbf{J}} & \frac{z_{12}}{\det \mathbf{J}} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix} \tag{6.25}$$

Potential-Energy Approach

The potential energy Π on the discretized region is given by

$$\Pi = \sum_e \left[\frac{1}{2}\left(2\pi \int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon} r\, dA\right) - 2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA - 2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{T} r\, d\ell\right] - \sum \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{6.26}$$

The element strain energy U_e given by the first term can be written as

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(2\pi \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} r\, dA\right)\mathbf{q} \tag{6.27}$$

The quantity inside the parentheses is the element stiffness matrix,

$$\mathbf{k}^e = 2\pi \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} r\, dA \tag{6.28}$$

The fourth row in $\mathbf{B}$ has terms of the type N_i/r. Further, this integral also has an additional r in it. As a simple approximation, $\mathbf{B}$ and r can be evaluated at the centroid of the triangle and used as representative values for the triangle. At the centroid of the triangle,

$$N_1 = N_2 = N_3 = \tfrac{1}{3} \tag{6.29}$$

and

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

where $\bar{r}$ is the radius of the centroid. Denoting $\bar{\mathbf{B}}$ as the element strain–displacement matrix $\mathbf{B}$ evaluated at the centroid, we get

$$\mathbf{k}^e = 2\pi\bar{r}\bar{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\mathbf{B}} \int_e dA$$

or

$$\mathbf{k}^e = 2\pi\bar{r}A_e\bar{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\bar{\mathbf{B}} \tag{6.30}$$

5. Applications/ Simulation/ related Laboratory example

The application of 3 noded axisymmetric triangular element is widely used in axisymmetric applications.

6. MCQ-Post test

1. How many nodes have in a Axisymmetric element
   - (a) 2
   - (b) 6
   - **(c) 3**
   - (d) 4
2. Axis-Symmetric element is______________Element
   - (a) 1D
   - **(b) 2D**
   - (c) 3D
   - (d) 4D
3. Axisymmetric triangular element has a stiffness matrix of order
   - (a) 2 x 2
   - (b) 4 x 4
   - **(c) 6 x 6**
   - (d) 1 x 1

7. Conclusion

The 3 noded axisymmetric element stiffness and body force terms were addressed.

8. References

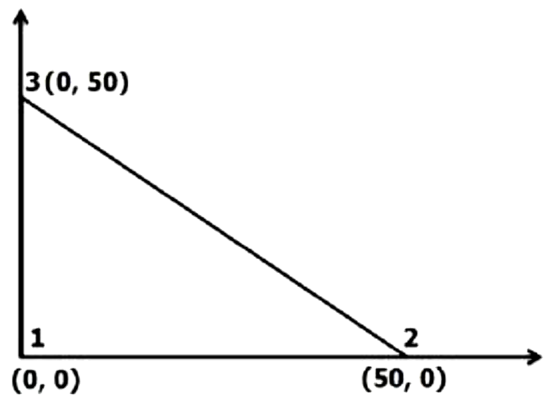
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=sZQ_pX_Lus4
https://www.youtube.com/watch?v=VAJ3-4iCvFw

10. Assignments

1. Determine the stiffness matrix for the axisymmetric element shown in figure. Take E as 2.1×10^6 N/mm^2 and Poisson's ratio as 0.3 . All dimensions are in mm.



UNIT-4

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Axisymmetric Continuum**

Name of the Topic : **Galarkin approach – Body forces and temperature effects**

1. Aim and Objectives

- To learn about on Galarkin approach and effects of body forces and temperature effects.

2. Pre-Test-MCQ type

1. The displacement function for r direction
   (a) N1u1-N2u2-N3u3
   (b) N1u1-N2u2+N3u3
   (c) N1u1+N2u2-N3u3
   (d) N1u1+N2u2+N3u3

2. From below, choose the correct condition for the axisymmetric element.
   (a) Symmetric about axis
   (b) Boundary conditions are symmetric about an axis
   (c) Loading conditions are symmetric about an axis
   (d) All the above

3. In FEA, the use of smaller-sized elements will lead to _______ computation time
   (a) less
   (b) more
   (c) depends on other factors
   (d) can't say

3. Prerequisites

- The engineering mathematics and engineering skill on axisymmetric domain required.

4. Theory behind

Galerkin Approach

In the Galerkin formulation, the consistent variation $\boldsymbol{\phi}$ in an element is expressed as

$$\boldsymbol{\phi} = \mathbf{N}\boldsymbol{\psi} \tag{6.42}$$

where

$$\boldsymbol{\psi} = [\psi_1, \psi_2, \ldots, \psi_6]^{\mathrm{T}} \tag{6.43}$$

The corresponding strain $\boldsymbol{\epsilon}(\phi)$ is given by

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \mathbf{B}\boldsymbol{\psi} \tag{6.44}$$

The global vector of variations $\boldsymbol{\Psi}$ is represented by

$$\boldsymbol{\Psi} = [\Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_N]^{\mathrm{T}} \tag{6.45}$$

We now introduce the interpolated displacements into the Galerkin variational form (Eq. 6.10). The first term representing the internal virtual work gives

$$\begin{aligned}\text{Internal virtual work} &= 2\pi \int_A \sigma^{\mathrm{T}}\boldsymbol{\epsilon}(\boldsymbol{\phi}) r\, dA \\ &= \sum_e 2\pi \int_e \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\boldsymbol{\psi} r\, dA \\ &= \sum_e \mathbf{q}^{\mathrm{T}}\mathbf{k}^e\boldsymbol{\psi}\end{aligned} \tag{6.46}$$

where the element stiffness $\mathbf{k}^e$ is given by

$$\mathbf{k}^e = 2\pi \bar{r} A_e \overline{\mathbf{B}}^{\mathrm{T}}\mathbf{D}\overline{\mathbf{B}} \tag{6.47}$$

We note that $\mathbf{k}^e$ is symmetric. Using the connectivity of the elements, the internal virtual work can be expressed in the form

$$\begin{aligned}\text{Internal virtual work} &= \sum_e \mathbf{q}^{\mathrm{T}}\mathbf{k}^e\boldsymbol{\psi} = \sum \boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \\ &= \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}\mathbf{Q}\end{aligned} \tag{6.48}$$

where $\mathbf{K}$ is the global stiffness matrix. The external virtual work terms in Eq. 6.10 involving body forces, surface tractions, and point loads can be treated in the same way as in the potential-energy approach, by replacing $\mathbf{q}$ with $\boldsymbol{\psi}$. The summation of all the force terms over the elements then yields

$$\text{External virtual work} = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{F} \tag{6.49}$$

Body Force Term

We first consider the body force term $2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA$. We have

$$\begin{aligned}2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA &= 2\pi \int_e (uf_r + wf_z) r\, dA \\ &= 2\pi \int_e [(N_1q_1 + N_2q_3 + N_3q_5)f_r + (N_1q_2 + N_2q_4 + N_3q_6)f_z] r\, dA\end{aligned}$$

Once again, approximating the variable quantities by their values at the centroid of the triangle, we get

$$2\pi \int_e \mathbf{u}^T \mathbf{f} r \, dA = \mathbf{q}^T \mathbf{f}^e \tag{6.32}$$

where the element body force vector $\mathbf{f}^e$ is given by

$$\mathbf{f}^e = \frac{2\pi \bar{r} A_e}{3} [\bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z]^T \tag{6.33}$$

The bar on the **f** terms indicates that they are evaluated at the centroid. Where body force is the primary load, greater accuracy may be obtained by substituting $r = N_1 r_1 + N_2 r_2 + N_3 r_3$ into Eq. 6.32 and integrating to get nodal loads.

Temperature Effects

Uniform increase in temperature of ΔT introduces initial normal strains $\boldsymbol{\epsilon}_0$ given as

$$\boldsymbol{\epsilon}_0 = [\alpha \Delta T, \quad \alpha \Delta T, \quad 0, \quad \alpha \Delta T]^T \tag{6.51}$$

The stresses are given by

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \tag{6.52}$$

where $\boldsymbol{\epsilon}$ is the total strain.

On substitution into the strain energy, this yields an additional term of $-\boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon}_0$ in the potential energy Π. Using the element strain–displacement relations in Eq. 6.24, we find that

$$2\pi \int_A \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon}_0 r \, dA = \sum_e \mathbf{q}^T (2\pi \bar{r} A_e \overline{\mathbf{B}}^T \mathbf{D} \bar{\boldsymbol{\epsilon}}_0) \tag{6.53}$$

The consideration of the temperature effect in the Galerkin approach is rather simple. The term $\boldsymbol{\epsilon}^T$ in Eq. (6.53) is replaced by $\boldsymbol{\epsilon}^T(\boldsymbol{\phi})$.

The expression in parentheses gives element nodal load contributions. The vector $\bar{\boldsymbol{\epsilon}}_0$ is the initial strain evaluated at the centroid, representing the average temperature rise of the element. We have

$$\boldsymbol{\Theta}^e = 2\pi \bar{r} A_e \overline{\mathbf{B}}^T \mathbf{D} \bar{\boldsymbol{\epsilon}}_0 \tag{6.54}$$

where

$$\boldsymbol{\Theta}^e = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6]^T \tag{6.55}$$

5. Applications/ Simulation/ related Laboratory example

- The application of 3 noded axisymmetric triangular element is widely used in axisymmetric applications.

6. MCQ-Post test

1. For axisymmetric element the strain is computed as

   (a) e =Bu

   (b) e =B-u

   (c) e =B+u

   (d) e =B/u

2. For thermal analysis, the field variable is __________.

   (a) stress

   (b) strain

   (c) displacement

   (d) temperature

3. The equation for thermal stress in each element is ________.

   (a) $\sigma = E\,(Bq + \alpha\,\Delta t)$

   (b) $\sigma = E\,(Bq - \alpha\,\Delta t)$

   (c) $\sigma = E\,(B + \alpha\,\Delta t)$

   (d) $\sigma = E\,(B - \alpha\,\Delta t)$

7. Conclusion

- The axisymmetric parameters are estimated by Galarkin approach and evaluation of body forces and temperature effects were studied.

8. References

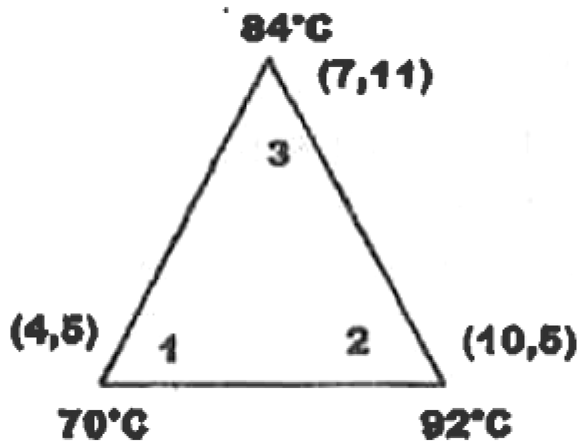
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=TZ7HQm7jPxk

10.Assignments

1. Nodal values of the triangular element is shown in Figure. Evaluate element shape functions and calculate he value of temperature at a point whose co-ordinates are given(5,7)



UNIT-4

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Axisymmetric Continuum**

Name of the Topic : **Stress calculations – Boundary conditions**

1. Aim and Objectives

- To calculate the element stress using the relevant Boundary conditions

2. Pre-Test-MCQ type

1. The triangular element stiffness metrics for axi-symmetric body is

   (a) $2 \pi r A B^T D B$

   (b) $2 \pi r A D B$

   (c) $2 \pi r B D B$

   (d) $\pi r A BT$

2. An circular section chimney with hot gases inside can be analyzed

   using model

   (a) full section

   (b) one half of section

   (c) one quarter of section

   (d) 1/8 th of section

3. Prerequisites

- A knowledge on axisymmetric domain required.

4. Theory behind

Stress Calculations

From the set of nodal displacements obtained above, the element nodal displacements **q** can be found using the connectivity. Then, using stress–strain relation in Eq. 6.8 and strain–displacement relation in Eq. 6.24, we have

$$\boldsymbol{\sigma} = \mathbf{D}\bar{\mathbf{B}}\mathbf{q} \qquad (6.50)$$

where $\bar{\mathbf{B}}$ is **B**, given in Eq. 6.25, evaluated at the centroid of the element. We also note that σ_θ is a principal stress. The two principal stresses σ_1 and σ_2 corresponding to σ_r, σ_z, and τ_{rz} can be calculated using Mohr's circle.

In Fig. E6.2, a long cylinder of inside diameter 80 mm and outside diameter 120 mm snugly fits in a hole over its full length. The cylinder is then subjected to an internal pressure of 2 MPa. Using two elements on the 10-mm length shown, find the displacements at the inner radius.

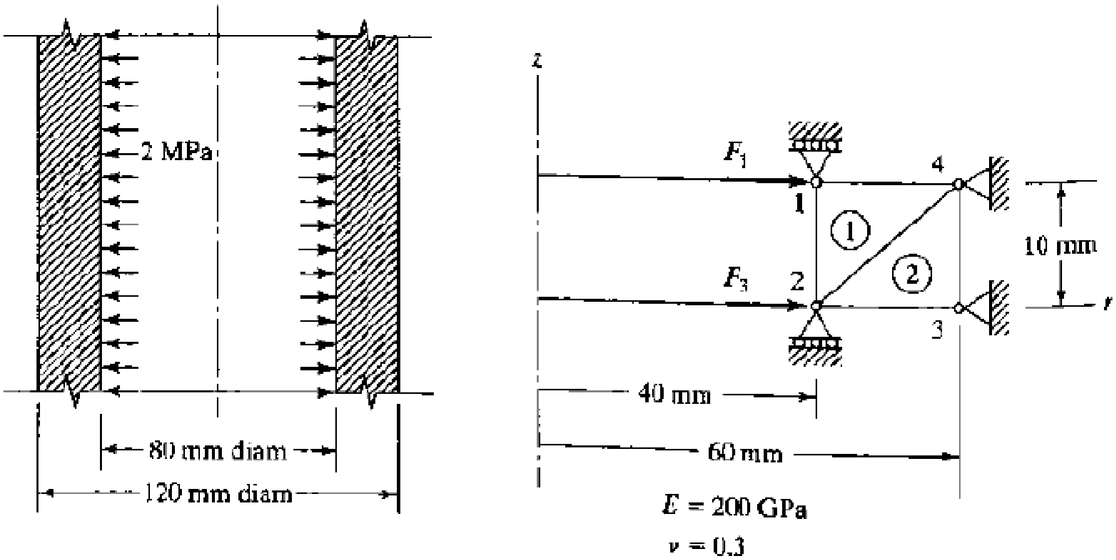


FIGURE E6.2

Example 6.3

Calculate the element stresses in the problem discussed in Example 6.2.

Solution We need to find $\sigma^{e^T} = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]^e$ for each element. From the connectivity established in Example 6.2,

$$\mathbf{q}^1 = [0.0140, \quad 0, \quad 0.0133, \quad 0, \quad 0, \quad 0]^T \times 10^{-2}$$
$$\mathbf{q}^2 = [0.0133, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]^T \times 10^{-2}$$

Using the product matrices $\mathbf{DB}^e$ and $\mathbf{q}$ in the formula

$$\boldsymbol{\sigma}^e = \mathbf{DB}^e\mathbf{q}$$

we get

$$\boldsymbol{\sigma}^1 = [-166, \quad -58.2, \quad 5.4, \quad -28.4]^T \times 10^{-2}\ \text{MPa}$$
$$\boldsymbol{\sigma}^2 = [-169.3, \quad -66.9, \quad 0, \quad -54.1]^T \times 10^{-2}\ \text{MPa}$$

5. Applications/ Simulation/ related Laboratory example

a. The various stress calculations on 3 noded axisymmetric triangular element is widely used in axisymmetric applications.

6. MCQ-Post test

1. Number of Stress components for 2D axi-symmetric element
   (a) 3
   (b) 4
   (c) 5
   (d) 6
2. Number of Strain components for 2D axi-symmetric element
   (a) 3
   (b) 6
   (c) 4
   (d) 5
3. In strain-displacement matrix[B], the mathematical value of β_2 is computed as
   (a) z3-z1
   (b) z3+z1
   (c) z3-z2
   (d) z1-z2

7. Conclusion

The element stress calculation for 3 noded axisymmetric element is effectively studied.

8. References

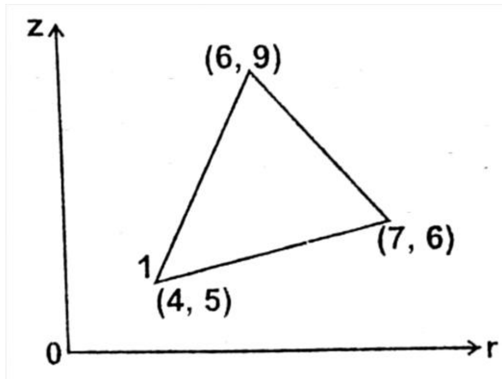
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=VAJ3-4iCvFw

10. Assignments

1. The Calculate the element stress for a triangular element shown in fig the nodal displacement are u_1= 0.001, u_2 = 0.002, u_3 = -0.003, w_1 = 0.002, w_2 = 0.001 and w_3 = 0.004 all dimensions are in mm.



UNIT-4

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : **Axisymmetric Continuum**

Name of the Topic : **Applications to cylinders under internal or external pressures – Rotating discs**

1. Aim and Objectives

- To encourage on studying of Axisymmetric based applications

2. Pre-Test-MCQ type

1. The co-ordinate of r is
   (a) (r1+r2+r3)/3
   (b) (r1-r2-r3)/3
   (c) (r1+r2-r3)/2
   (d) (r1-r2+r3)/3
2. The value of z coordinate is found by the which relationship
   (a) (z1-z2-z3)/3
   (b) (z1+z2+z3)/2
   (c) (z1+z2+z3)/3
   (d) (z1-z2+z3)/3

3. Prerequisites

- The engineering skill on axisymmetric domain required.

4. Theory behind

We have seen that the axisymmetric problem simply reduces to consideration of the revolving area. The boundary conditions need to be enforced on this area. θ independence arrests the rotation. Axisymmetry also implies that points lying on the z-axis remain radially fixed. Let us now consider some typical problems with a view to modeling them.

Cylinder Subjected to Internal Pressure

Figure 6.7 shows a hollow cylinder of length L subjected to an internal pressure. One end of the cylindrical pipe is attached to a rigid wall. In this, we need to model only the rectangular region of the length L bound between r_i and r_0. Nodes on the fixed end are constrained in the z and r directions. Stiffness and force modifications will be made for these nodes.

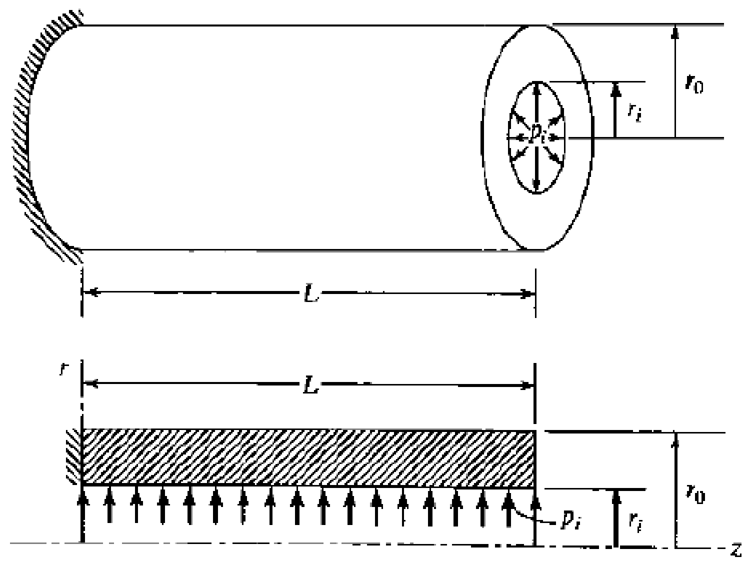


FIGURE 6.7 Hollow cylinder under internal pressure.

Infinite Cylinder

In Fig. 6.8, modeling of a cylinder of infinite length subjected to external pressure is shown. The length dimensions are assumed to remain constant. This plane strain condition is modeled by considering a unit length and restraining the end surfaces in the z direction.

Press Fit on a Rigid Shaft

Press fit of a ring of length L and internal radius r_i onto a rigid shaft of radius $r_i + \delta$ is considered in Fig. 6.9. When symmetry is assumed about the midplane, this plane is restrained in the z direction. When we impose the condition that nodes at the internal radius have to displace radially by δ, a large stiffness C is added to the diagonal locations for the radially constrained dofs and a force $C\delta$ is added to the corresponding force components. Solution of the equations gives displacements at nodes; stresses can then be evaluated.

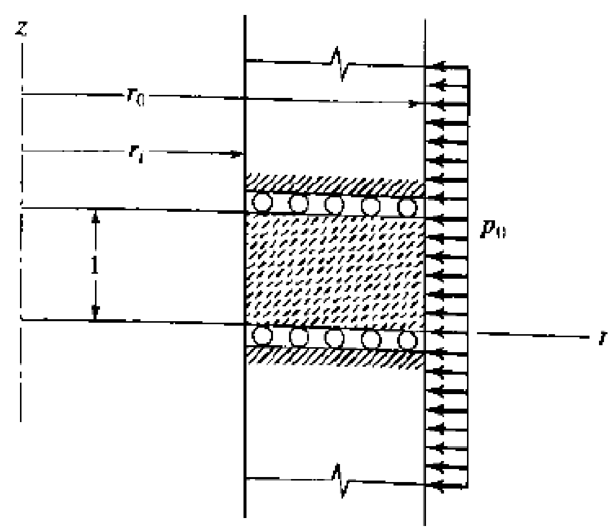
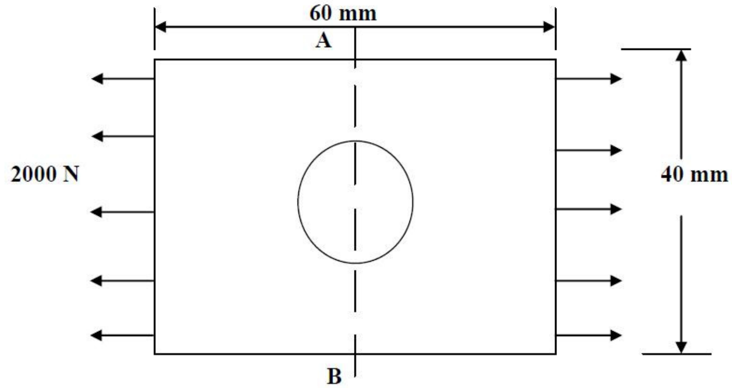


FIGURE 6.8 Cylinder of infinite length under external pressure.

5. Applications/ Simulation/ related Laboratory example

In the plate with a hole under plane stress, find deformed shape of the hole and determine the maximum stress distribution along A-B (you may use t = 1 mm). E = 210GPa, t = 1 mm, Poisson's ratio = 0.3, Dia of the circle = 10 mm, Analysis assumption plane stress with thickness is used.



6. MCQ-Post test

1. The order of stress components for axisymmetric element
   (a) Radial stress, Longitudinal stress, Circumferential stress, shear stress
   (b) Radial stress, Longitudinal stress, Circumferential stress, shear stress
   (c) Radial stress, Circumferential stress, shear stress, Longitudinal stress
   (d) Longitudinal stress, Radial stress, Circumferential stress, shear stress

2. The stress calculations for axisymmetric element by the following relation
   (a) σ=DB/u
   (b) σ=DB-u
   (c) σ=DBu
   (d) σ= - DBu

3. Open ended thin cylinder is considered as
   (a) Axisymmetric
   (b) Plane strain
   (c) Plane stress
   (d) All of the these

7. Conclusion

The various applications of Axisymmetric applications were studied.

8. References

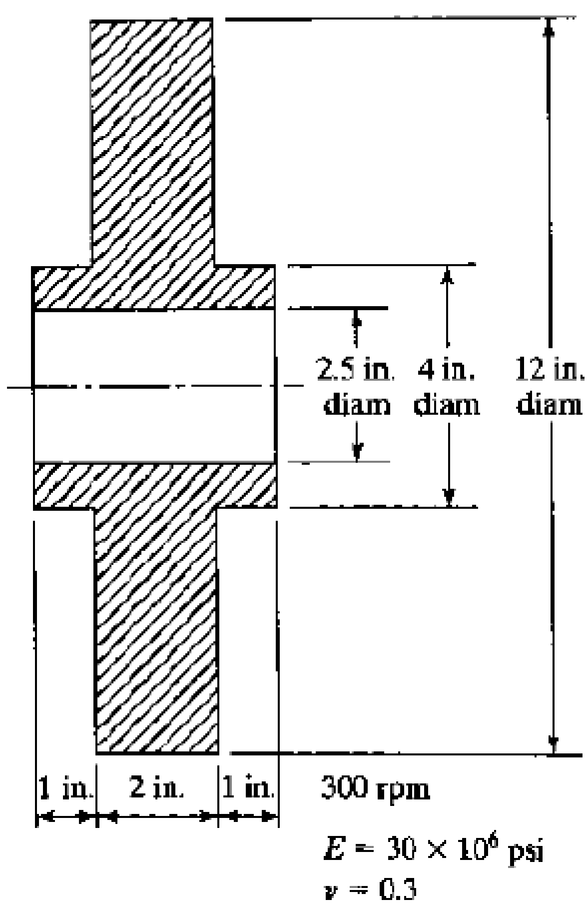
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., “Introduction to Finite Elements in Engineering”, Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON “Fundamentals of Finite Element Analysis”2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. “Finite Element Analysis”, New Age International Publishers, 2005, India
- RAO S.S., “The Finite Element Method in Engineering”, Pergammon Press, 1989
- P.SESHU “Textbook of Finite Element Analysis”, PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=WlWh-BNmnAc
https://www.youtube.com/watch?v=AJa8szV0FM8

10. Assignments

The steel flywheel shown in Figure rotates at 3000 rpm. Find the deformed shape of the flywheel and give the stress distribution.



UNIT-5

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : Isoparametric elements for two dimensional continnum

Name of the Topic : **The four node quadrilateral – Shape functions**

1. Aim and Objectives

- To understand the Isoparametric elements for two dimensional Continnum
- To acquire the effectiveness of shape function for isoparametric element

2. Pre-Test-MCQ type

1. From the following, which type of element is not two dimensional?
   (a) Rectangle
   (b) Quadrilateral
   (c) Parallelogram
   (d) Tetrahedron

2. The finite element method is mostly used in the field of
   (a) structural mechanics
   (b) classical mechanics
   (c) applied mechanics
   (d) engineering mechanics

3. Finite element analysis deals with ____________ .
   (a) approximate numerical solution
   (b) non-boundary value problems
   (c) partial differential equations
   (d) laplace equations

3. Prerequisites

- The engineering mathematics and complex boundary based theoretical knowledge required.

4. Theory behind

7.2 THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 7.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and (x_i, y_i) are the coordinates of node i. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^T$ denotes the element displacement vector. The displacement of an interior point P located at (x, y) is represented as $\mathbf{u} = [u(x, y), v(x, y)]^T$.

Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 7.2. The master element is defined in ξ-, η-coordinates (or *natural* coordinates) and is square shaped. The Lagrange shape functions where $i = 1, 2, 3$, and 4, are defined such that N_i is equal to unity at node i and is zero at other nodes. In particular, consider the definition of N_i:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{7.1}$$

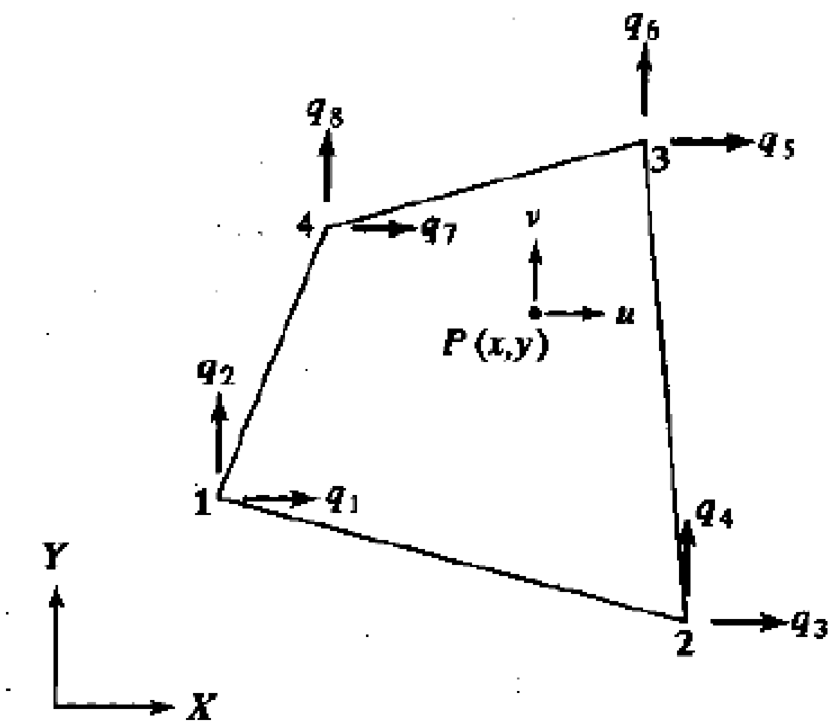


FIGURE 7.1 Four-node quadrilateral element.

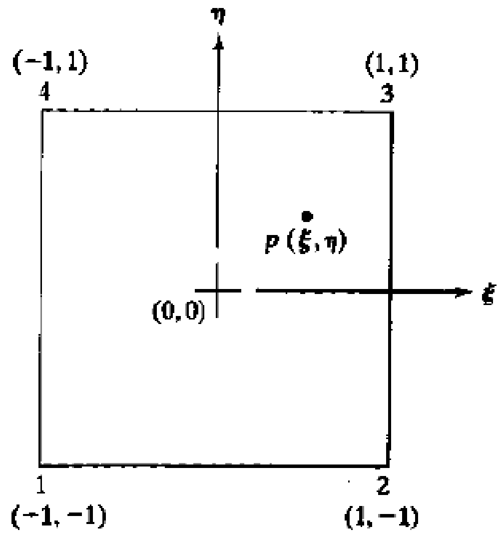


FIGURE 7.2 The quadrilateral element in ξ, η space (the *master* element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along edges $\xi = +1$ and $\eta = +1$ (Fig. 7.2). Thus, N_1 has to be of the form

$$N_1 = c(1 - \xi)(1 - \eta) \tag{7.2}$$

where c is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1, \eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{7.3}$$

which yields $c = \frac{1}{4}$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{7.4}$$

All the four shape functions can be written as

$$\begin{aligned} N_1 &= \tfrac{1}{4}(1 - \xi)(1 - \eta) \\ N_2 &= \tfrac{1}{4}(1 + \xi)(1 - \eta) \\ N_3 &= \tfrac{1}{4}(1 + \xi)(1 + \eta) \\ N_4 &= \tfrac{1}{4}(1 - \xi)(1 + \eta) \end{aligned} \tag{7.5}$$

While implementing in a computer program, the compact representation of Eqs. 7.5 is useful

$$N_i = \tfrac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i) \tag{7.6}$$

where (ξ_i, η_i) are the coordinates of node i.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^{\mathrm{T}}$ represents the displacement components of a point located at (ξ, η), and $\mathbf{q}$, dimension (8×1), is the element displacement vector, then

$$u = N_1q_1 + N_2q_3 + N_3q_5 + N_4q_7$$
$$v = N_1q_2 + N_2q_4 + N_3q_6 + N_4q_8 \tag{7.7a}$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{7.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{7.8}$$

In the isoparametric formulation, we use the *same* shape functions N_i to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$
$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \tag{7.9}$$

Subsequently, we will need to express the derivatives of a function in x-, y-coordinates in terms of its derivatives in ξ-, η-coordinates. This is done as follows: A function $f = f(x, y)$, in view of Eqs. 7.9, can be considered to be an implicit function of ξ and η as $f = f[x(\xi, \eta), y(\xi, \eta)]$. Using the chain rule of differentiation, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \tag{7.10}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{7.11}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{7.12}$$

In view of Eqs. 7.5 and 7.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{7.13a}$$

$$\equiv \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{7.13b}$$

Equation 7.11 can be inverted as

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14a}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\,dy = \det \mathbf{J}\, d\xi\, d\eta \tag{7.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

5. Applications/ Simulation/ related Laboratory example

- The shape functions for 4 noded isoparametric element conceptually used in irregular boundaries of the domain.

6. MCQ-Post test

1. Curved boundary is better modeled by using
   (a) non-dimensional shape functions
   (b) higher order elements
   (c) more number of simple elements
   (d) isoparametric elements
2. When fewer nodes are used to define the geometry than are used to define the displacement, the element is called
   (a) subparametric element
   (b) isoparametric element
   (c) superparametric element
   (d) complex element
3. When same number of nodes are used to define the geometry and displacement, the element is called
   (a) subparametric element
   (b) isoparametric element
   (c) superparametric element
   (d) simple element

7. Conclusion

The formulation of isoparametric elements for irregular boundaries and 4 noded quadrilateral element is deeply studied.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
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- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

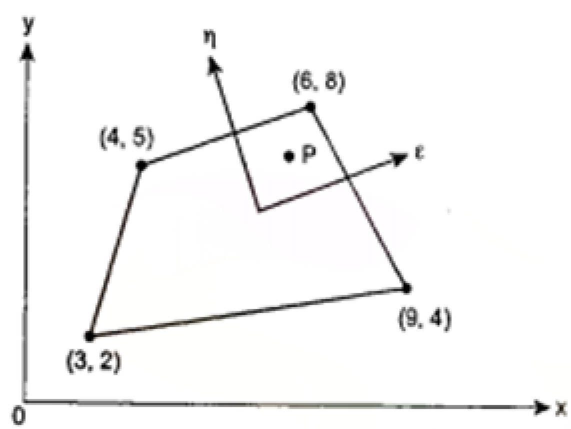
9. Video

https://www.youtube.com/watch?v=4lc7SYxJ9F8

https://www.youtube.com/watch?v=PhedVyx_G8o

10.Assignments

1. Evaluate the Cartesian co-ordinate of the point P which has local co-oedinates ε= 0.6 and η=0.8 as shown in Figure.



UNIT-5

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : Isoparametric elements for two dimensional continnum

Name of the Topic : **Element stiffness matrix and force vector**

1. Aim and Objectives

- To assess the element stiffness matrix and force vector for isoparametric element

2. Pre-Test-MCQ type

1. What is a matrix?
   (a) Group of elements
   (b) Array of elements
   (c) Group of columns and rows
   (d) Array of numbers

2. The vector $q=[q_1,q_2 \ldots\ldots\ldots q_8]T$ of a four noded quadrilateral denotes
   (a) Load vector
   (b) Transition matrix
   (c) Element displacement vector
   (d) Constant matrix

3. Prerequisites

- The basics of matrices and the irregular boundary based knowledge domain is required.

4. Theory behind

Element Stiffness Matrix

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \tfrac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\epsilon}\, dV \tag{7.16}$$

or

$$U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\epsilon}\, dA \tag{7.17}$$

where t_e is the thickness of element e.

The strain–displacement relations are

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{7.18}$$

By considering $f \equiv u$ in Eq. 7.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{7.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.19b}$$

Equations 7.18 and 7.19a,b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{7.21}$$

Now, from the interpolation equations Eqs. 7.7a, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{Gq} \tag{7.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{7.23}$$

Equations 7.20 and 7.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{7.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{7.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{7.26}$$

where $\mathbf{D}$ is a (3×3) material matrix. The strain energy in Eq. 7.17 becomes

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det \mathbf{J}\, d\xi\, d\eta\right]\mathbf{q} \tag{7.27a}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{7.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det \mathbf{J}\, d\xi\, d\eta} \tag{7.28}$$

is the element stiffness matrix of dimension (8×8).

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. (7.28) are involved functions of ξ and η, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

Element Force Vectors

Body Force A body force that is distributed force per unit volume, contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential-energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV \tag{7.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ as constant within each element, we get

$$\int_V \mathbf{u}^T \mathbf{f}\, dV = \sum_e \mathbf{q}^T \mathbf{f}^e \tag{7.30}$$

where the (8×1) element body force vector is given by

$$\mathbf{f}^e = t_e \left[\int_{-1}^{1} \int_{-1}^{1} \mathbf{N}^T \det \mathbf{J}\, d\xi\, d\eta \right] \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \tag{7.31}$$

As with the stiffness matrix derived earlier, this body force vector has to be evaluated by numerical integration.

5. Applications/ Simulation/ related Laboratory example

- The main applications of element stiffness matrix and force vectors are useful for determining the nodal displacements, stress and strains.

6. MCQ-Post test

1. The Jacobian matrix is a
   (a) single column matrix
   (b) diagonal matrix
   (c) matrix of any dimension
   (d) square matrix
2. The shape function at Node 1 for four node rectangular element as
   (a) N1=0.52(1-ξ) (1-η)
   (b) N1=0.52(1+ξ) (1+η)
   (c) N1=0.25(1+ξ) (1-η)
   (d) N1=0.25(1-ξ) (1-η)
3. The value of J_{11} in the Jacobian Matrix is represented by
   (a) $J_{11}=0.25[-(1-\eta)x_1+(1-\eta)\, x_2+(1-\eta)\, x_3-(1+\eta)\, x_3]$
   (b) $J_{11}=0.5[-(1-\eta)x_1+(1-\eta)\, x_2+(1+\eta)\, x_3-(1+\eta)\, x_3]$
   (c) $J_{11}=0.2\,[-(1-\eta)x_1-(1-\eta)\, x_2+(1+\eta)\, x_3-(1+\eta)\, x_3]$
   (d) $J_{11}=0.25[-(1-\eta)x_1+(1-\eta)\, x_2+(1+\eta)\, x_3-(1+\eta)\, x_3]$
4. Iso-Parametric Element is _____Element
   (a) Regular
   (b) Ir-regular
   (c) Sub
   (d) Super
5. Nodal points greater than geometry points is known as___________
   (a) Isoparametric
   (b) Subparametric
   (c) Superperametric
   (d) CST

6. Conclusion

The 4 noded quadrilateral element stiffness matrix and load vector were studied well.

7. References

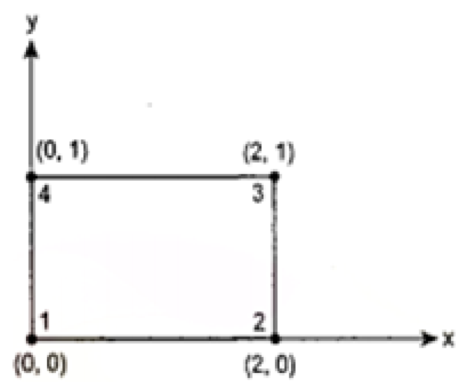
- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

8. Video

https://www.youtube.com/watch?v=6JKG2EvvlfA
https://www.youtube.com/watch?v=PhedVyx_G8o

9. Assignments

1. A four node rectangular element is shown in Figure. Determine the following: (a) Jacobian Matrix; (b) Starin- Displacement matrix and (c) Element stresses.



UNIT-5

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : Isoparametric elements for two dimensional continnum

Name of the Topic : **Element stiffness matrix and force vector**

1. Aim and Objectives

- To Evaluate the element stiffness matrix and force vector

2. Pre-Test-MCQ type

1. Evaluate the integral $\int \frac{(10x^9 + 10^x \log_e 10)}{x^{10} + 10^x} dx$

(a) $10^x + x^{10} + C$

(b) $10^x - x^{10} + C$

(c) $\log_{10}(10^x + x^{10}) + C$

(d) $\log_e(10^x + x^{10}) + C$

2. Integration of function is same as the ___________

(a) Joining many small entities to create a large entity
(b) Indefinitely small difference of a function
(c) Multiplication of two function with very small change in value
(d) Point where function neither have maximum value nor minimum value

3. Prerequisites

- The basics of integration knowledge is required.

4. Theory behind

NUMERICAL INTEGRATION

Consider the problem of numerically evaluating a one-dimensional integral of the form

$$I = \int_{-1}^{1} f(\xi)\, d\xi \qquad (7.33)$$

The *Gaussian quadrature* approach for evaluating I is given subsequently. This method has proved most useful in finite element work. Extension to integrals in two and three dimensions follows readily.

Consider the n-point approximation

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) + \cdots + w_n f(\xi_n) \qquad (7.34)$$

where $w_1, w_2, \ldots,$ and w_n are the **weights** and $\xi_1, \xi_2, \ldots,$ and ξ_n are the sampling points or **Gauss points**. The idea behind Gaussian quadrature is to select the n Gauss points and n weights such that Eq. 7.34 provides an exact answer for polynomials $f(\xi)$ of as large a degree as possible. In other words, the idea is that if the n-point integration formula is exact for all polynomials up to as high a degree as possible, then the formula will work well even if f is not a polynomial. To get some intuition for the method, the one-point and two-point approximations are discussed in the sections that follow.

One-Point Formula. Consider the formula with $n = 1$ as

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) \tag{7.35}$$

Since there are two parameters, w_1 and ξ_1, we consider requiring the formula in Eq. 7.35 to be exact when $f(\xi)$ is a polynomial of order 1. Thus, if $f(\xi) = a_0 + a_1\xi$, then we require

$$\text{Error} = \int_{-1}^{1} (a_0 + a_1\xi)\, d\xi - w_1 f(\xi_1) = 0 \tag{7.36a}$$

$$\text{Error} = 2a_0 - w_1(a_0 + a_1\xi_1) = 0 \tag{7.36b}$$

or

$$\text{Error} = a_0(2 - w_1) - w_1 a_1 \xi_1 = 0 \tag{7.36c}$$

From Eq. 7.36c, we see that the error is zeroed if

$$w_1 = 2 \qquad \xi_1 = 0 \tag{7.37}$$

For any general f, then, we have

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx 2f(0) \tag{7.38}$$

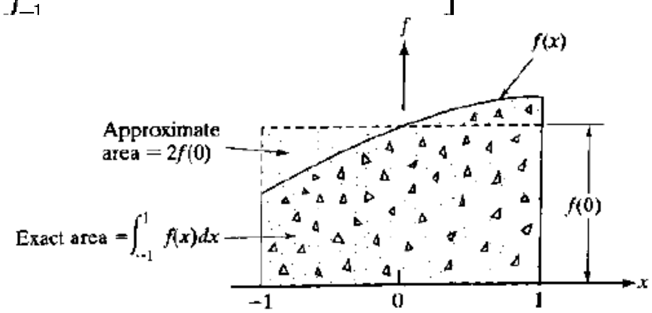
which is seen to be the familiar *midpoint rule* (Fig. 7.3).

Two-Point Formula. Consider the formula with $n = 2$ as

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) \tag{7.39}$$

We have four parameters to choose: $w_1, w_2, \xi_1,$ and ξ_2. We can therefore expect the formula in Eq. 7.39 to be exact for a cubic polynomial. Thus, choosing $f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$ yields

$$\text{Error} = \left[\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3)\, d\xi\right] - [w_1 f(\xi_1) + w_2 f(\xi_2)] \tag{7.40}$$



Requiring zero error yields

$$\begin{aligned} w_1 + w_2 &= 2 \\ w_1\xi_1 + w_2\xi_2 &= 0 \\ w_1\xi_1^2 + w_2\xi_2^2 &= \tfrac{2}{3} \\ w_1\xi_1^3 + w_2\xi_2^3 &= 0 \end{aligned} \tag{7.41}$$

These nonlinear equations have the unique solution

$$w_1 = w_2 = 1 \qquad -\xi_1 = \xi_2 = 1/\sqrt{3} = 0.5773502691\ldots \tag{7.42}$$

From this solution, we can conclude that n-point Gaussian quadrature will provide an exact answer if f is a polynomial of order $(2n - 1)$ or less. Table 7.1 gives the values of w_i and ξ_i for Gauss quadrature formulas of orders $n = 1$ through $n = 6$. Note that the Gauss points are located symmetrically with respect to the origin and that symmetrically placed points have the same weights. Moreover, the large number of digits given in Table 7.1 should be used in the calculations for accuracy (i.e., use double precision on the computer).

TABLE 7.1 Gauss Points and Weights for Gaussian Quadrature

$$\int_{-1}^{1} f(\xi)\,d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i)$$

Number of points, n	Location, ξ_i	Weights, w_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	±0.7745966692	0.5555555556
	0.0	0.8888888889
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549
5	±0.9061798459	0.2369268851
	±0.5384693101	0.4786286705
	0.0	0.5688888889
6	±0.9324695142	0.1713244924
	±0.6612093865	0.3607615730
	±0.2386191861	0.4679139346

5. Applications/ Simulation/ related Laboratory example

- The Gaussian quadrature approach used in many numerical method applications

6. MCQ-Post test

1. In one point gauss quadrature problems, the w_i is

(a) 2.0
(b) 1.5
(c) 1.8
(d) 2.9

2.In gauss quadrature problems, the $f(x_i)$ is

(a) values of the function at pre-determined sampling points

(b) values of the function at post-determined sampling points

(c) values of the function at post-determined nodal points

(d) values of the function at pre-determined nodal points

7. Conclusion

- The Numerical integration using gauss quadrature effectively studied.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=17w-NbjysCc

https://www.youtube.com/watch?v=El0o_vCHL7Y

10.Assignments

1. Evaluate the integral by using 3 point Gaussion Quadrature $\int_{-1}^{1}(x^3+2x^2+4x)dx$.
2. Evaluate the integral, $I=\int_{-1}^{1}[x^2+\cos(x/2)]dx$ using three point Gaussian Quadrature and compare with exact solutions.
3. Integrate the function $f(r)=1+r+r^2+r^3$ between the limits -1 and +1 using,
   - (i) Exact method
   - (ii) Gauss integration method and compare the two results.

UNIT-5

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : Isoparametric elements for two dimensional continnum

Name of the Topic : **Stiffness integration – Stress calculations**

1. Aim and Objectives

- An attempt to study on stiffness integration and stress calculations

2. Pre-Test-MCQ type

1. Evaluate the integral of dx / (x + 2) from -6 to -10.

   (a) 21/2

   (b) 1/2

   (c) ln 3

   (d) ln 2

2. What is the integral of sin5 x cos3 x dx if the lower limit is zero and the upper limit is π/2?

   (a) 0.0203

   (b) 0.0307

   (c) 0.0417

   (d) 0.0543

3. Prerequisites

a. The engineering mathematics and engineering skill on axisymmetric domain required.

4. Theory behind

Stiffness Integration

To illustrate the use of Eq. 7.44, consider the element stiffness for a quadrilateral element

$$\mathbf{k}^e = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B} \det \mathbf{J} \, d\xi \, d\eta$$

where **B** and det **J** are functions of ξ and η. Note that this integral actually consists of the integral of each element in an (8 × 8) matrix. However, using the fact that $\mathbf{k}^e$ is symmetric, we do not need to integrate elements below the main diagonal.

Let ϕ represent the ijth element in the integrand. That is, let

$$\phi(\xi, \eta) = t_e(\mathbf{B}^T\mathbf{D}\mathbf{B} \det \mathbf{J})_{ij} \tag{7.45}$$

Then, if we use a 2 × 2 rule, we get

$$k_{ij} \approx w_1^2\phi(\xi_1, \eta_1) + w_1w_2\phi(\xi_1, \eta_2) + w_2w_1\phi(\xi_2, \eta_1) + w_2^2\phi(\eta_2, \eta_2) \tag{7.46a}$$

where $w_1 = w_2 = 1.0, \xi_1 = \eta_1 = -0.57735\ldots$, and $\xi_2 = \eta_2 = +0.57735\ldots$. The Gauss points for the two-point rule used above are shown in Fig. 7.4. Alternatively, if we label the Gauss points as 1, 2, 3, and 4, then k_{ij} in Eq. 7.46a can also be written as

$$k_{ij} = \sum_{\mathrm{IP}=1}^{4} W_{\mathrm{IP}}\phi_{\mathrm{IP}} \tag{7.46b}$$

where ϕ_{IP} is the value of ϕ and W_{IP} is the weight factor at integration point IP. We note that $W_{\mathrm{IP}} = (1)(1) = 1$. Computer implementation is sometimes easier using Eq. 7.46b.

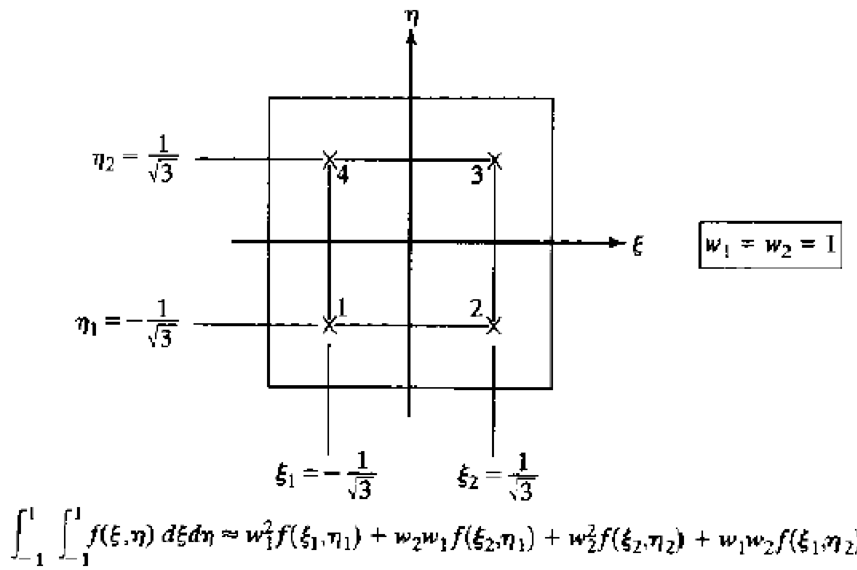


FIGURE 7.4 Gaussian quadrature in two dimensions using the 2×2 rule.

We may readily follow the implementation of the previous integration procedure in program QUAD provided at the end of this chapter.

The evaluation of three-dimensional integrals is similar. For triangles, however, the weights and Gauss points are different, as discussed later in this chapter.

Stress Calculations

Unlike the constant-strain triangular element (Chapters 5 and 6), the stresses $\boldsymbol{\sigma} = \mathbf{DBq}$ in the quadrilateral element are not constant within the element; they are functions of ξ and η, and consequently vary within the element. In practice, the stresses are evaluated at the Gauss points, which are also the points used for numerical evaluation of $\mathbf{k}^e$, where they are found to be accurate. For a quadrilateral with 2×2 integration, this gives four sets of stress values. For generating less data, one may evaluate stresses at one point per element, say, at $\xi = 0$ and $\eta = 0$. The latter approach is used in the program QUAD.

Example 7.2

Consider a rectangular element as shown in Fig. E7.1. Assume plane stress condition, $E = 30 \times 10^6$ psi, $\nu = 0.3$, and $\mathbf{q} = [0, 0, 0.002, 0.003, 0.006, 0.0032, 0, 0]^T$ in. Evaluate $\mathbf{J}$, $\mathbf{B}$, and $\boldsymbol{\sigma}$ at $\xi = 0$ and $\eta = 0$.

Solution Referring to Eq. 7.13a, we have

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} 2(1-\eta) + 2(1+\eta) & (1+\eta) - (1+\eta) \\ -2(1+\xi) + 2(1+\xi) & (1+\xi) + (1-\xi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

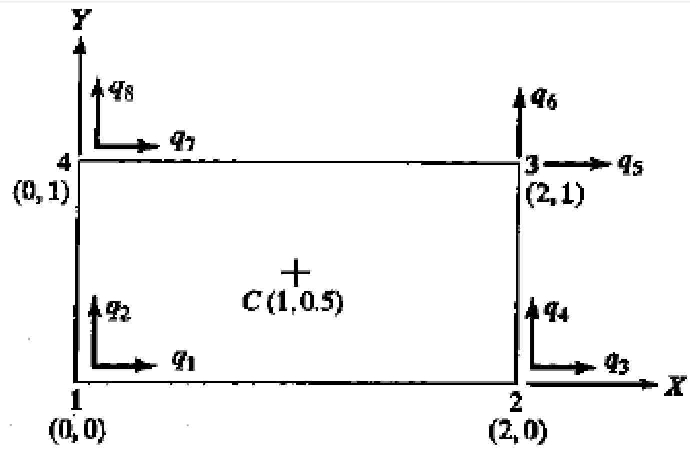


FIGURE E7.1

For this rectangular element, we find that $\mathbf{J}$ is a constant matrix. Now, from Eqs. 7.21,

$$\mathbf{A} = \frac{1}{1/2}\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Evaluating $\mathbf{G}$ in Eq. 7.23 at $\xi = \eta = 0$ and using $\mathbf{B} = \mathbf{QG}$, we get

$$\mathbf{B}^0 = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

The stresses at $\xi = \eta = 0$ are now given by the product

$$\boldsymbol{\sigma}^0 = \mathbf{DB}^0\mathbf{q}$$

For the given data, we have

$$\mathbf{D} = \frac{30 \times 10^6}{(1 - 0.09)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.03 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Thus,

$$\boldsymbol{\sigma}^0 = [66\,920.\ 23\,080.\ 40\,960]^T \text{ psi}$$

5. Applications/ Simulation/ related Laboratory example

- The stiffness integration and stress calculations more useful for solving irregular boundary problems

6. MCQ-Post test

1. Gaussian points are used for
   (a) Numerical integration
   (b) displacement calculation
   (c) Stress calculation
   (d) strain calculation

2. In 3 point gauss quadrature , the assumed weights values (w1 and w2) are
   (a) 0.888888 and 0.555555
   (b) 5.555555 and 8.888888
   (c) 0.888888 and 0.444444
   (d) 0.555555 and 0.888888

3. Conclusion

- The stiffness integration and stress calculations are studied..

4. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
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- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

5. Video

https://www.youtube.com/watch?v=3He0rE5Arrs

6. Assignments

1. Write short notes on stiffness integration for isoparametric elements.

UNIT-5

Name of the Course : **FINITE ELEMENT ANALYSIS (FEA)**

Name of the Unit : Isoparametric elements for two dimensional continnum

Name of the Topic : **Four node quadrilateral for axisymmetric problems**

1. Aim and Objectives

- To study about Four node quadrilateral for axisymmetric problems

2. Pre-Test-MCQ type

1. Which of terms referred as local co-ordinates
   (a) ξ and η
   (b) x and y
   (c) r and z
   (d) none of these
2. Cartesian co-ordinates generally originated by
   (a) x,y,θ
   (b) u,v
   (c) x,y
   (d) None of these

3. Prerequisites

- The engineering mathematics and engineering skill on axisymmetric domain required.

4. Theory behind

FOUR-NODE QUADRILATERAL FOR AXISYMMETRIC PROBLEMS

The stiffness development for the four node-quadrilateral for axisymmetric problems follows steps similar to the quadrilateral element presented earlier. The x-, y-coordinates are replaced by r, z. The main difference occurs in the development of the $\mathbf{B}$ matrix, which relates the four strains to element nodal displacements. We partition the strain vector as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \epsilon_\theta \end{bmatrix} \tag{7.59}$$

where $\bar{\boldsymbol{\epsilon}} = [\epsilon_r \; \epsilon_z \; \gamma_{rz}]^{\mathrm{T}}$.

Now in the relation $\boldsymbol{\epsilon} = \mathbf{Bq}$, we partition $\mathbf{B}$ as $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$, such that $\mathbf{B}_1$ is a 3×8 matrix relating $\bar{\boldsymbol{\epsilon}}$ and $\mathbf{q}$ by

$$\bar{\boldsymbol{\epsilon}} = \mathbf{B}_1\mathbf{q} \tag{7.60}$$

and $\mathbf{B}_2$ is a row vector 1 × 8 relating ϵ_θ and $\mathbf{q}$ by

$$\epsilon_\theta = \mathbf{B}_2\mathbf{q} \tag{7.61}$$

Noting that r, z replace x, y, it is clear that $\mathbf{B}_1$ is same as the 3 × 8 matrix given in Eq. 7.24 for the four-node quadrilateral. Since $\epsilon_\theta = u/r$ and $u = N_1q_1 + N_1q + N_3q_3 + N_4q_4$, $\mathbf{B}_2$ can be written as

$$\mathbf{B}_2 = \left[\frac{N_1}{r} \quad 0 \quad \frac{N_2}{r} \quad 0 \quad \frac{N_3}{r} \quad 0 \quad \frac{N_4}{r} \quad 0\right] \tag{7.62}$$

On introducing these changes, the element stiffness is then obtained by performing numerical integration on

$$\mathbf{k}^e = 2\pi \int_{-1}^{1}\int_{-1}^{1} r\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta \tag{7.63}$$

The force terms (in Eq. 7.31 and 7.32) are to be multiplied by the factor of 2π as in the axisymmetric triangle.

The axisymmetric quadrilateral element has been implemented in the program AXIQUAD.

5. Applications/ Simulation/ related Laboratory example

- The isoparametric based axisymmetric triangular element is widely used in axisymmetric applications.

6. MCQ-Post test

1. The matrix dimension of element stiffness matrix for 4 node quadrilateral element as specified by
   (a) 8 x8
   (b) 7 x7
   (c) 6 x6
   (d) 5 x5

2. The Cartesian co-ordinate x for isoparametric quadrilateral element is obtained by
   (a) $N_1y_1+N_2x_2-N_3x_3+N_4x_4$
   (b) $N_1x_1-N_2y_2+N_3x_3+N_4x_4$
   (c) $N_1x_1-N_2x_2-N_3x_3-N_4x_4$
   (d) $N_1x_1+N_2x_2+N_3x_3+N_4x_4$

3. The Jacobian matrix is a

(a) single column matrix
(b) diagonal matrix
(c) matrix of any dimension
(d) square matrix

7. Conclusion

- The application of axisymmetric triangular element (isoparametric) for irregular boundary of axisymmetric condition is studied.

8. References

- CHANDRUPATLA T.R., AND BELEGUNDU A.D., "Introduction to Finite Elements in Engineering", Pearson Education 2002, 3rd Edition.
- DAVID V HUTTON "Fundamentals of Finite Element Analysis"2004. McGraw-Hill Int. Ed.
- BHAVIKATTI S.S. "Finite Element Analysis", New Age International Publishers, 2005, India
- RAO S.S., "The Finite Element Method in Engineering", Pergammon Press, 1989
- P.SESHU "Textbook of Finite Element Analysis", PHI Learning Private Limited, India.

9. Video

https://www.youtube.com/watch?v=q4uENUY_oV8

10. Assignments

1. Write short notes with suitable axisymmetric condition using Four node quadrilateral element.

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