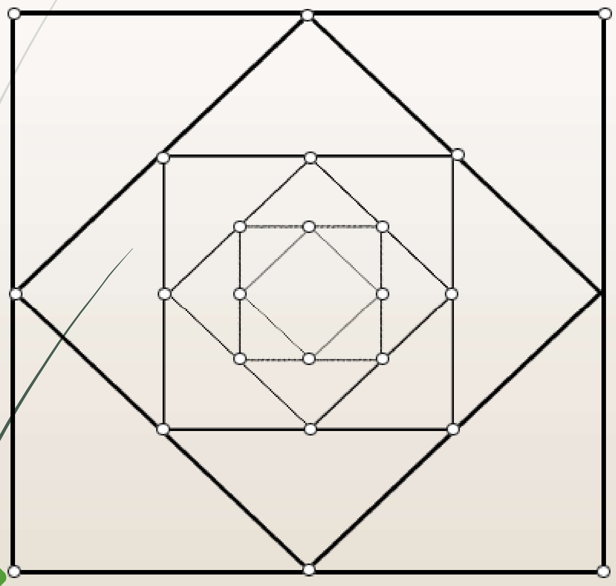
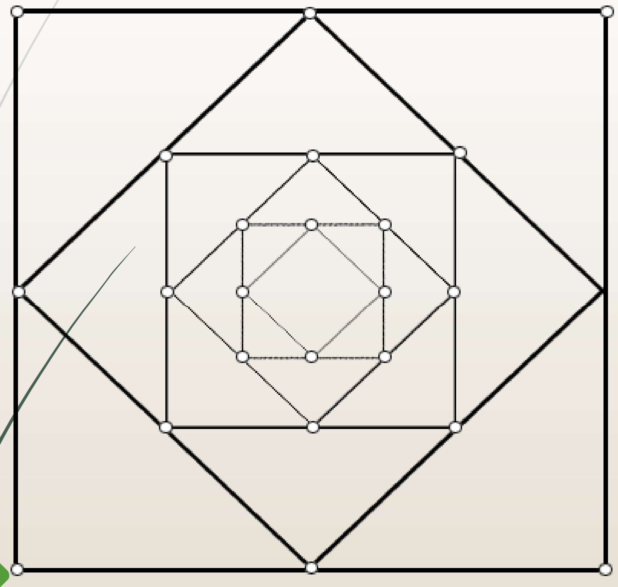


Graph Theory: Eulerian Graph



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Department of Mathematics

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- Lecture I
- Unit IV
- Eulerian and Hamiltonian Graphs
- Graph Theory
- IV Semester
- II B.Sc Mathematics

Eulerian Graph - Outline

- **Aim**
- **Learning Outcome**
- **Prerequisites**
- **Preliminaries**
- **Konigsberg Bridge Problem**
- **Definitions**
- **Euler’s Theorem**
- **Fleury’s Algorithm**

Aim

- Clarify the basic concept of path and circuits
- Explain about Euler path and circuit
- Fleury’s algorithm

Learning Outcome

- **Students should be able to do identify Euler circuits and paths**
- **To understand the history of Konigsberg Bridge Problem**

Prerequisites

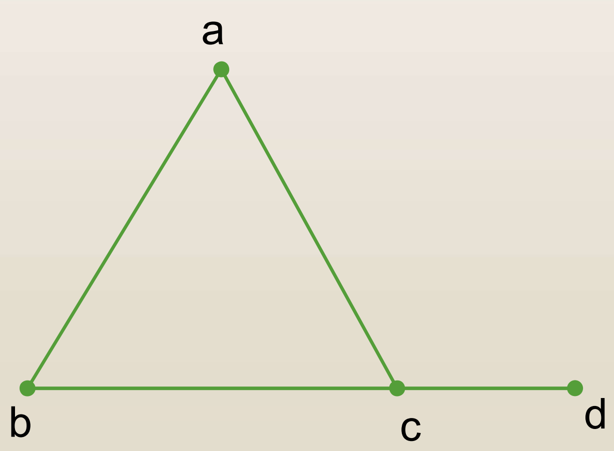
- **Some elementary knowledge of linear algebra would be helpful. In addition, a general experience in mathematics.**

Preliminaries

- A *graph* is a pair G=(V,E) of sets satisfying

$$E \subseteq [V]^2.$$

  - The elements of *V* are the vertices of our graph.
  - The elements of *E* are the edges of our graph.



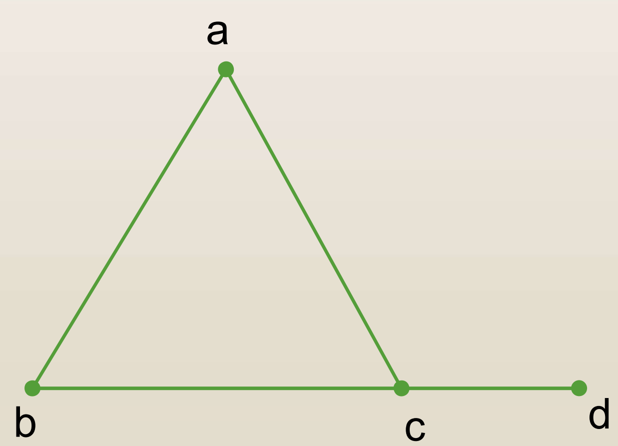
V={a,b,c,d}

E={{a,b},{a,c},{b,c},{c,d}}

Preliminaries

- The *degree* of a vertex v (denoted deg(v)) is the number of edges directly connected to that vertex.

deg(a)=2 deg(b)=2
deg(c)=3deg(d)=1

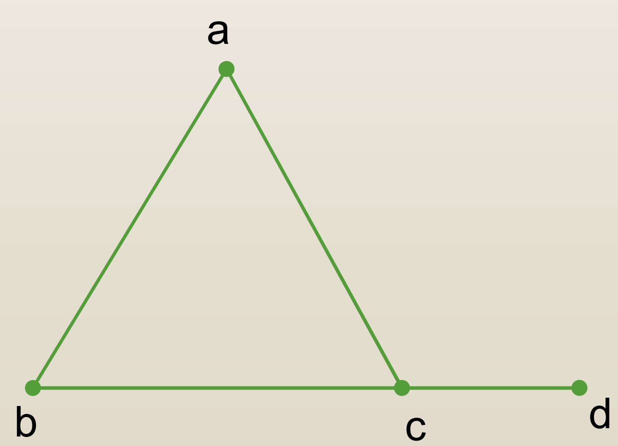


V={a,b,c,d}

E={{a,b},{a,c},{b,c},{c,d}}

Preliminaries

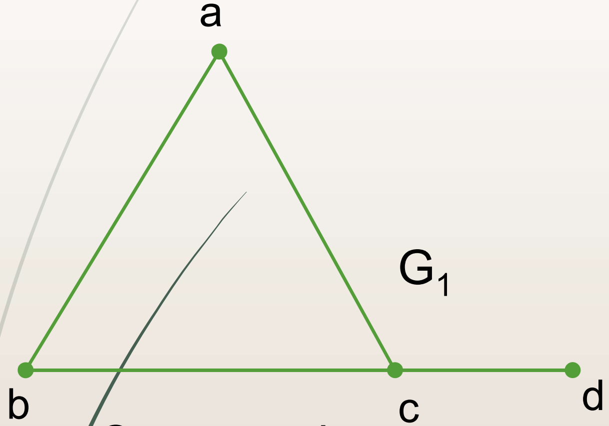
- A *path* of length n>0 is a sequence of edges that begins at a vertex of the graph and travels from vertex to vertex along the edges of a graph.
- If a path begins and ends at the same vertex, it is called a *closed path* or *circuit*.



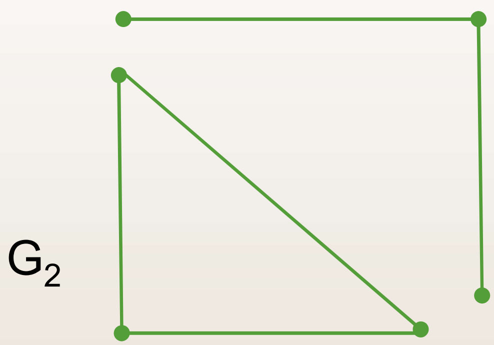
e.g.
a,b,c,d is a path
a,b,c,a is a circuit.

Preliminaries

- A graph is *connected* if it cannot be expressed as the union of two graphs



Connected



Disconnected

In other words, a graph is connected if there is a path between every distinct vertex of the graph.

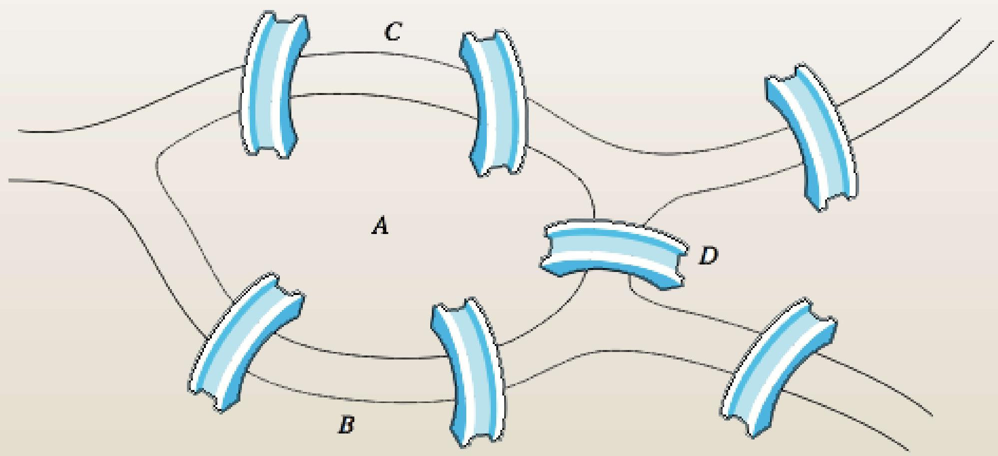
Konigsberg Bridge Problem

In the early 1700s, the city of Konigsberg was the capital of East Prussia. The river Pregel ran through the city in two branches with an island between the branches (see figure on next slide).

There were seven bridges joining various parts of the city. The following problem was well known. Is it possible for a citizen of Konigsberg to take a stroll through the city, crossing each bridge exactly once, and beginning and ending at the same place?

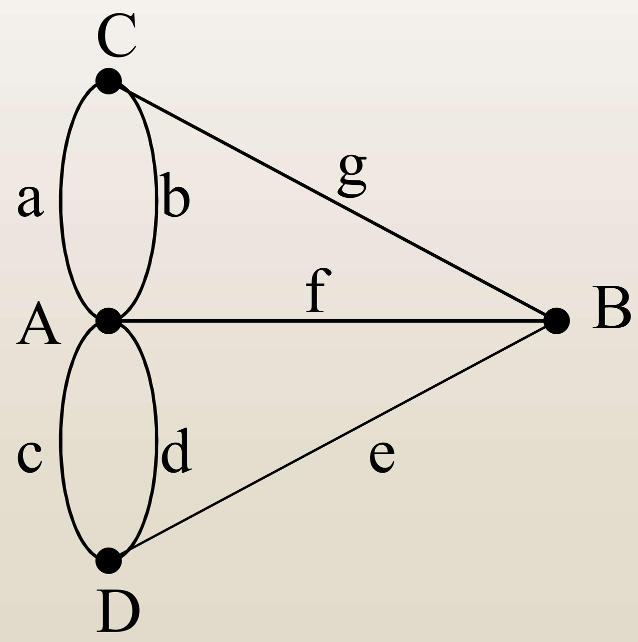
The Seven Bridges of Königsberg

Question: Is it possible to start at some location in town, travel across all seven bridges without crossing any bridge twice, and return to the same starting point?

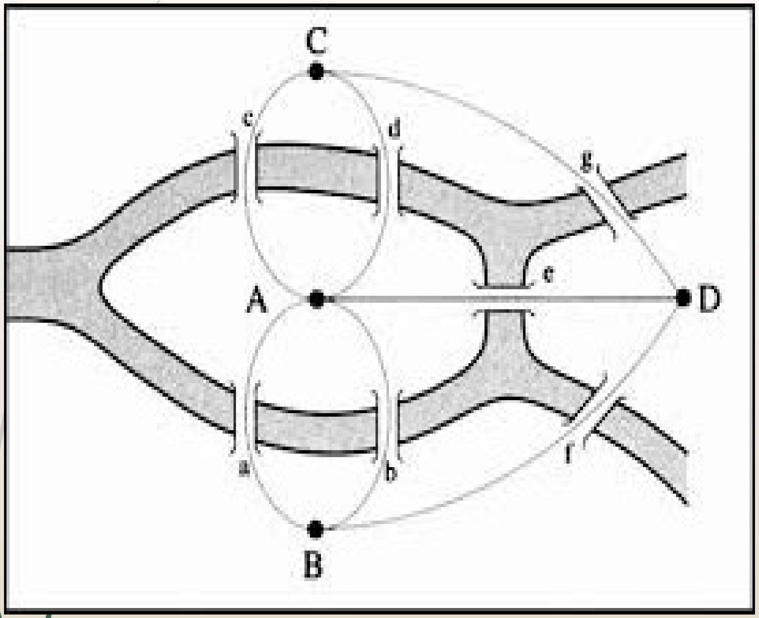


Graph of the Map

The vertices represent landmasses. The edges represent the bridges



Solution for Koingsberg problem

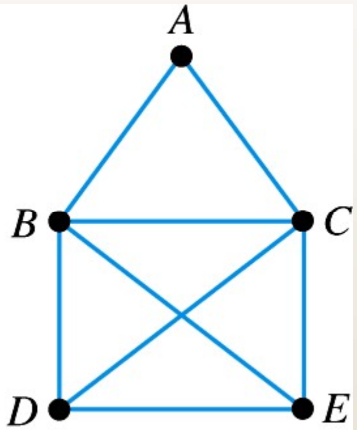


Definitions

- An **Euler path** is a path that passes through each edge of a graph exactly one time.
- An **Euler circuit** is a circuit that passes through each edge of a graph exactly one time.
- The difference between an Euler path and an Euler circuit is that an Euler circuit must start and end at the same vertex.

Examples

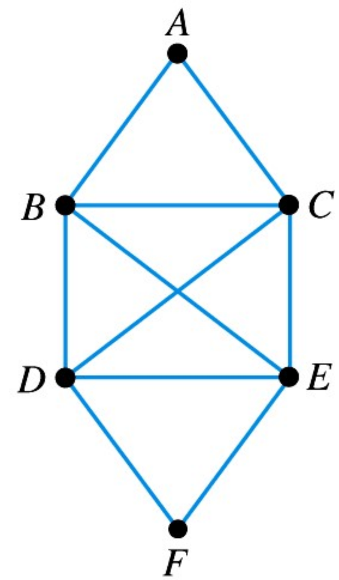
- Euler path



(a)

An Euler path
D, E, B, C, A, B, D, C, E

- Euler circuit

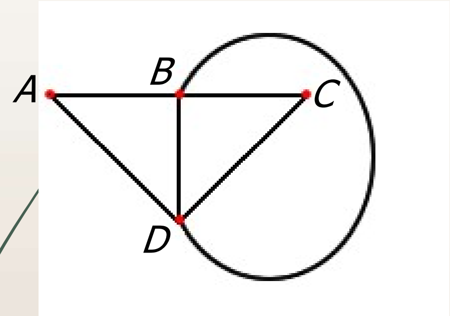


(b)

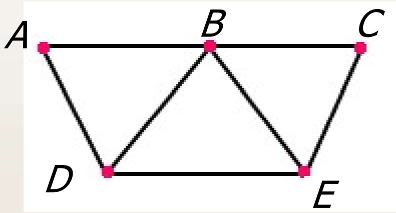
An Euler circuit
D, E, B, C, A, B, D, C, E, F, D

Example: Euler Path and Circuits

- For the graphs shown, determine if an Euler path, an Euler circuit, neither, or both exist.



The graph has many Euler circuits, each of which is also an Euler path. This graph has no odd vertices. One example is *A, D, B, C, D, B, A*.



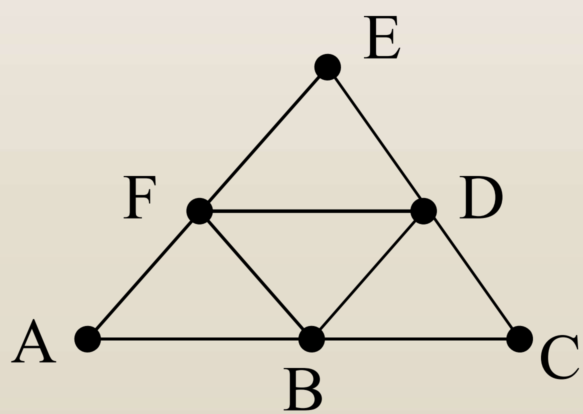
The graph has an Euler path but it does not have an Euler circuit. One Euler path is *E, C, B, E, D, B, A, D*. Each path must begin or end at vertex *D* or *E*.

Example: Recognizing Euler Circuits

Consider the graph below.

a) Is $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow A$ an Euler circuit?

b) Does the graph have an Euler circuit?

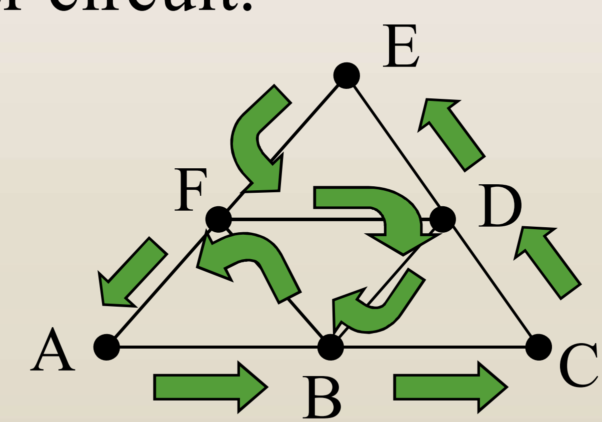


Example: Recognizing Euler Circuits

Solution

a) No, it does not use edge BD.

b) Yes, the circuit
$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow D \rightarrow B \rightarrow F \rightarrow A$
is an Euler circuit.



Euler's Theorem

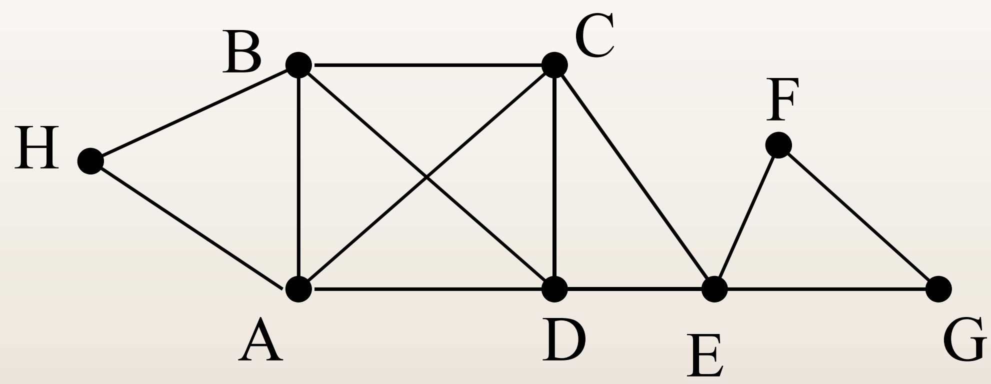
Suppose we have a connected graph.

1. If the graph has an Euler circuit, then each vertex of the graph has even degree.

2. If each vertex of the graph has even degree, then the graph has an Euler circuit.

Example: Using Euler's Theorem

Does the graph below have an Euler circuit?



Solution

Yes, because the graph is connected and each vertex has even degree.

Fleury's Algorithm

- *Fleury's algorithm* can be used to find an Euler circuit in any connected graph in which each vertex has even degree.
- An algorithm is like a recipe; follow the steps and you achieve what you need.

Fleury's Algorithm

Step 1:

Start at any vertex. Go along any edge from this vertex to another vertex. *Remove this edge from the graph.*

Step 2:

You are now on a vertex on the revised graph. Choose any edge from this vertex, but not a cut edge, unless you have no other option. Go along your chosen edge. *Remove this edge from the graph.*

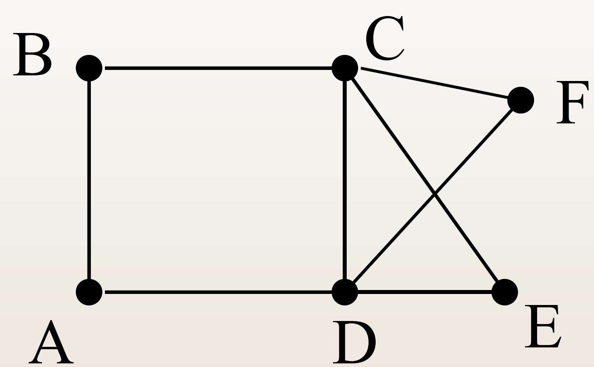
Fleury's Algorithm

Step 3:

Repeat Step 2 until you have used all the edges and gotten back to the vertex at which you started.

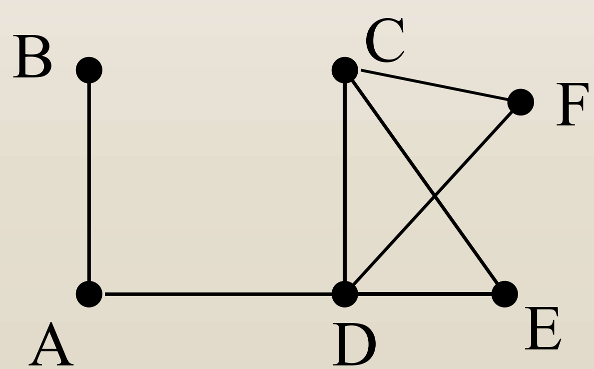
Example: Using Fleury's Algorithm

Find an Euler circuit for the graph below.



Solution

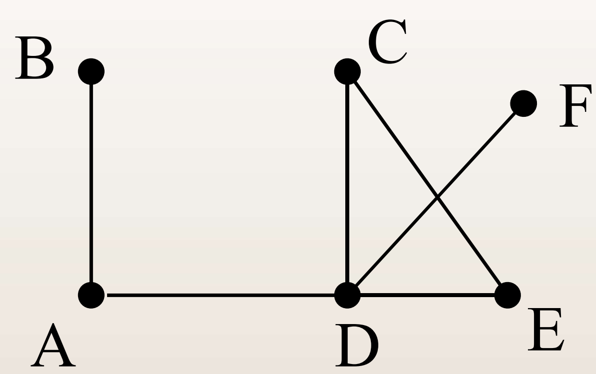
Remove BC



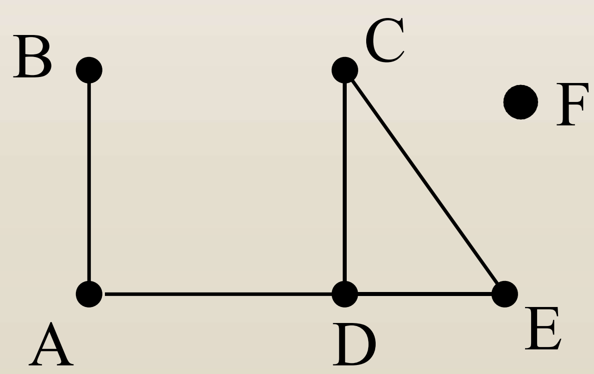
Example: Using Fleury's Algorithm

Solution (continued)

Remove
CF



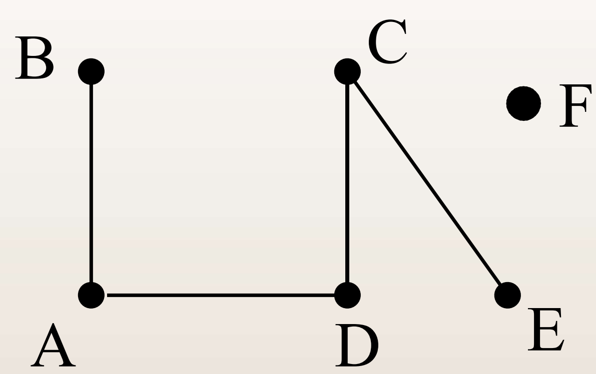
Remove
FD



Example: Using Fleury's Algorithm

Solution (continued)

Remove
DE



Now it is clear to finish with
$E \rightarrow C \rightarrow D \rightarrow A \rightarrow B.$

Example: Using Fleury's Algorithm

Solution (continued)

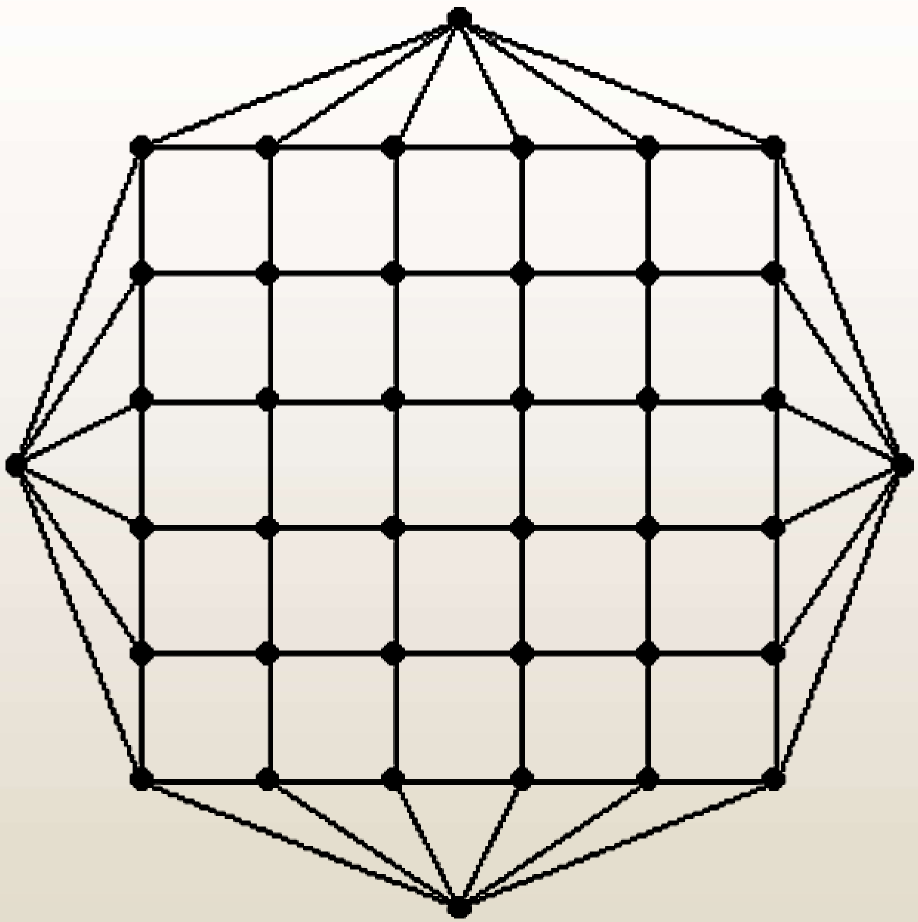
The complete Euler circuit is

$$B \to C \to F \to D \to E \to C \to D \to A \to B.$$

Note that a graph that has an Euler circuit always has more than one Euler circuit.

Thank You

- References
  - R. Diestel, Graph Theory.
  - L.R. Foulds, Graph Theory Applications.
  - K. Rosen, Discrete Mathematics and Its Applications.
  - R. Wilson, Introduction To Graph Theory.



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