I BSC MATHS

Unit V

TRIGNOMETRY

Topic 1 : Expansions of cosnθ and sinnθ

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Objectives:

The formulation of the formula to determine the coefficients of cosθ and sinθ. The general formula of the expansion of cos nθ and sin nθ is obtain from the formulas of the coefficients.

Pre-requesties:

- De Moviers Theorem

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

- Binomial Theorem

$$(x+a)^n = x^n + n_{c_1} x^{n-1} a + n_{c_2} x^{n-2} a^2 + \ldots + a^n$$

- Compound identities

Expansions of $\cos n\theta$ and $\sin n\theta$

Demovier's Theorem is

$$\left(\cos n\theta + i\sin n\theta\right) = \left(\cos\theta + i\sin\theta\right)^n$$

If n is a positive integer, the expansion on the right-hand side can be expanded by Binomial Theorem.

Binomial Theorem is $(x+a)^n = x^n + n_{c1}x^{n-1}a + n_{c2}x^{n-2}a^2 + \ldots + n_{cn}a^n$

$$\left(\cos n\theta + i\sin n\theta\right) = \cos^n\theta + n\cos^{n-1}\theta(i\sin\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta(i$$

$$\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta(i\sin\theta)^3 + \ldots$$

$$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \ldots$$

Hence we have

$$\left(\cos n\theta + i\sin n\theta\right) = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-}{4!}$$

$$+\ldots + i\left(n\cos^{n-1}\theta\sin\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-}\right.$$

Equating the real and imaginary parts we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\text{co}$$

-…

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \ldots$$

Remark:

1. The terms are alternately positive and negative
2. Each series continues till one of the factors in the numerator is zero and then ceases.
3. The sum of the powers of $\cos\theta$ and $\sin\theta$ in every term of the expansions equals n.

Both the series are in descending powers of $\cos\theta$ and in ascending powers of $\sin\theta$

Corollory:

1. $$\frac{\sin n\theta}{\sin\theta} = n\cos^{n-1}\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^2\theta + \ldots$$

$$= n\cos^{n-1}\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta(1-\cos^2\theta) +$$

$$.\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta(1-\cos^2\theta)^2 + \ldots$$

Similarly in the expansions of $\cos n\theta$, by putting

$\sin^2\theta = 1-\cos^2\theta$,

$\cos n\theta$ can be expressed in a series containing powers of $\cos\theta$.

2. Coefficient of $\cos^{n-1}\theta$ in the expansion of

$$\frac{\sin n\theta}{\sin\theta} = n_{c_1} + n_{c_3} + n_{c_5} + \ldots = 2^{n-1}$$

3. Coefficient of $\cos^{n}\theta$ in the expansion of

$$\cos n\theta = n_{c_0} + n_{c_2} + n_{c_4} + \ldots = 2^{n-1}$$

Expansion of $\tan n\theta$ **in powers of** $\tan\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$\tan n\theta = \frac{\cos^n\theta + n_{c_2}\cos^{n-2}\theta\sin^2\theta + n_{c_4}\cos^{n-4}\theta\sin^4\theta + \ldots}{n\cos^{n-1}\theta\sin\theta + n_{c_3}\cos^{n-3}\theta\sin^3\theta + \ldots}$$

On dividing both the numerator and denominator by $\cos^n\theta$

Expansion of $\tan(A + B + C + ...)$

$$\cos A + i\sin A = \cos A(1 + i\tan A)$$

$$\cos B + i\sin B = \cos B(1 + i\tan B)$$

$$\cos C + i\sin C = \cos C(1 + i\tan C)$$

$$\therefore (\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)...$$
$$= \cos A\cos B\cos C...(1 + i\tan A)(1 + i\tan B)(1 + i\tan C)...$$

$$= \cos A \cos B \cos C \ldots \left[1 + i\sum \tan A + i^2 \sum \tan A \tan B + i^3 \sum \tan A \tan B\right.$$

$$= \cos A \cos B \cos C \ldots [1 + iS_1 - S_2 - iS_3 + \ldots]$$

Where S_r is the sum of products taken r at a time of tan A, tan B, tan C,…

Equating the real and imaginary parts on both sides, we have

$$\cos(A + B + C + \ldots) = \cos A \cos B \cos C \ldots (1 - S_2 + S_4 + \ldots)$$

$$\sin(A + B + C + \ldots) = \cos A \cos B \cos C \ldots (S_1 - S_3 + S_5 + \ldots)$$

$$\therefore \tan(A + B + C + \ldots) = \frac{S_1 - S_3 + S_5 + \ldots}{1 - S_2 + S_4 + \ldots}$$

Corollary

Putting $A = B = C = ... = \theta$ taking n angles

Where S_r is the sum of the products taken r at a time of tan A, tan A, …, tan A n terms

Hence $S_1 = n\tan\theta$ $S_2 = n_{c_2}\tan^2\theta$, $S_3 = n_{c_3}\tan^3\theta, \ldots$

$$\tan n\theta = \frac{n_{c_1}\tan\theta - n_{c_2}\tan^3\theta + ...}{1 - n_{c_2}\tan^2\theta + n_{c_4}\tan^4\theta + ...}$$

Examples:

1) Express cos8θ in terms of sinθ

Solution

$$\left(\cos 8\theta + i\sin 8\theta\right) = \left(\cos\theta + i\sin\theta\right)^8$$

$$= \cos^8\theta + 8_{c_1}\cos^7\theta(i\sin\theta) + 8_{c_2}\cos^6\theta(i\sin\theta)^2 + \ldots$$

$$= \cos^8\theta - 8_{c_2}\cos^6\theta\sin^2\theta + 8_{c_4}\cos^4\theta\sin^4\theta - 8_{c_6}\cos^2\theta\sin^6\theta + 8_{c_8}$$

$$i(8_{c_1}\cos^7\theta\sin\theta + 8_{c_3}\cos^5\theta\sin^3\theta + 8_{c_5}\cos^3\theta\sin^5\theta + 8_{c_7}\cos\theta\sin^7\theta)$$

Equating the real parts, we have

$$\cos 8\theta = \cos^8\theta - 8_{c_2}\cos^6\theta\sin^2\theta +$$
$$8_{c_4}\cos^4\theta\sin^4\theta - 8_{c_6}\cos^2\theta\sin^6\theta + 8_{c_8}\sin^8\theta.$$

$$\cos 8\theta = \left(1-\sin^2\theta\right)^4 - 28\left(1-\sin^2\theta\right)^3\sin^2\theta + 70\left(1-\sin^2\theta\right)^2\sin^4\theta$$
$$-28\left(1-\sin^2\theta\right)\sin^6\theta + \sin^8\theta.$$

$$\cos 8\theta = \left(1-\sin^2\theta\right)^4 - 28\left(1-\sin^2\theta\right)^3\sin^2\theta + 70\left(1-\sin^2\theta\right)^2\sin^4\theta$$
$$-28\left(1-\sin^2\theta\right)\sin^6\theta + \sin^8\theta.$$

$$= (1-4\quad\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta) - 28(1-3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin^2\theta\,\cdot$$
$$70(1-2\sin^2\theta + \sin^4\theta)\sin^4\theta - 28(1-\sin^2\theta)\sin^6\theta + \sin^8\theta$$
$$=\quad(1+28+70+28+1)\sin^8\theta + (-4-84-140-28)\sin^6\theta + (6+84+70)\sin^4\theta$$
$$(-4-28)\sin^2\theta + 1$$
$$\cos 8\theta = 128\sin^8\theta - 256\sin^6\theta + 160\sin^4\theta - 32\sin^2\theta + 1$$

2) Express $\dfrac{\sin 6\theta}{\sin\theta}$ in terms of $\cos\theta$

Solution:

$$\left(\cos 6\theta + i\sin 6\theta\right) = \left(\cos\theta + i\sin\theta\right)^6$$

$$= \cos^6\theta + 6_{c_1}\cos^5\theta(i\sin\theta) + 6_{c_2}\cos^4\theta(i\sin\theta)^2 + 6_{c_3}\cos^3\theta(i\sin\theta$$

$$6_{c_5}\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$$

$$= \cos^6\theta + 6_{c_2}\cos^4\theta\sin^2\theta + 6_{c_4}\cos^2\theta\sin^4\theta - \sin^6\theta +$$

$$i\left(6_{c_1}\cos^5\theta\sin\theta - 6_{c_3}\cos^3\theta\sin^3\theta 6_{c_5}\cos\theta\sin^5\theta\ \right)$$

Equating the imaginary parts on both sides,

$$\sin 6\theta = 6_{c_1}\cos^5\theta\sin\theta - 6_{c_3}\cos^3\theta\sin^3\theta + 6_{c_5}\cos\theta\sin^5\theta$$

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta$$

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

Exercises:

1. Write down the expansions of

i) $\sin 6\theta$

ii) $\cos 5\theta$

iii) $\cos 9\theta$

iv) $\tan 4\theta$

v) $\tan 9\theta$

2. Expand $\sin 7\theta$ in powers of $\cos\theta$ and $\sin\theta$. Deduce that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

3. Prove that

$$\frac{\sin 9\theta}{\sin\theta} = 256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 49\cos^2\theta + 1$$

Applications of Trigonometry:

- Engineering, physics, surveying, architecture, astronomy and even in the investigation of a crime scene.

Apart from astronomy and geography,

- Trigonometry is applicable in various fields like

satellite navigation, developing computer music, chemistry number theory, medical imaging, electronics, electrical engineering, civil engineering, architecture, mechanical engineering, oceanography, seismology, phonetics, image compression and game development