Green's Theorem

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jectives

this section Student 's will learn the following :

Green's theorem which connects the line integral with the double integral.

Computations of line integrals and area enclosed by a curve.

To present the fundamental concepts of multivariable calculus and to develo

student understanding

Skills in the topic necessary for its applications to Science and Engineering.

Learning outcomes

Upon completion of this course, students should be able to

- ❖ Integrate functions of several variables over curves and surfaces.
- ❖ Use Green's theorem to compute integrals.
- ❖ Communicate Calculus and other Mathematical ideas effectively in speech and in writing.

Relationship to Program Learning Outcomes (PLOs)

The successful completion of this course will increase student' s

- Solve mathematical problems using analytical methods.
- Recognize the relationships between different areas of mathematics and the connections between mathematics and other disciplines.
- Give clear and organized written and verbal explanations_of mathematical ideas to a variety of students.

Introduction

reen's theorem is simply a relationship between the macroscopic circulation aroun rve C and the sum of all the microscopic circulation that is inside C.

C is a simple closed curve in the plane (remember, we are talking about mensions), then it surrounds some region D in the plane.

reen's theorem relates a line integral around a simply closed plane curve C and a d tegral over the region enclosed by C.

he theorem is useful because it allows us to translate difficult line integrals into mple double integrals, or difficult double integrals into more simple line integral.

finition:

nsider a region R in a plane defined by

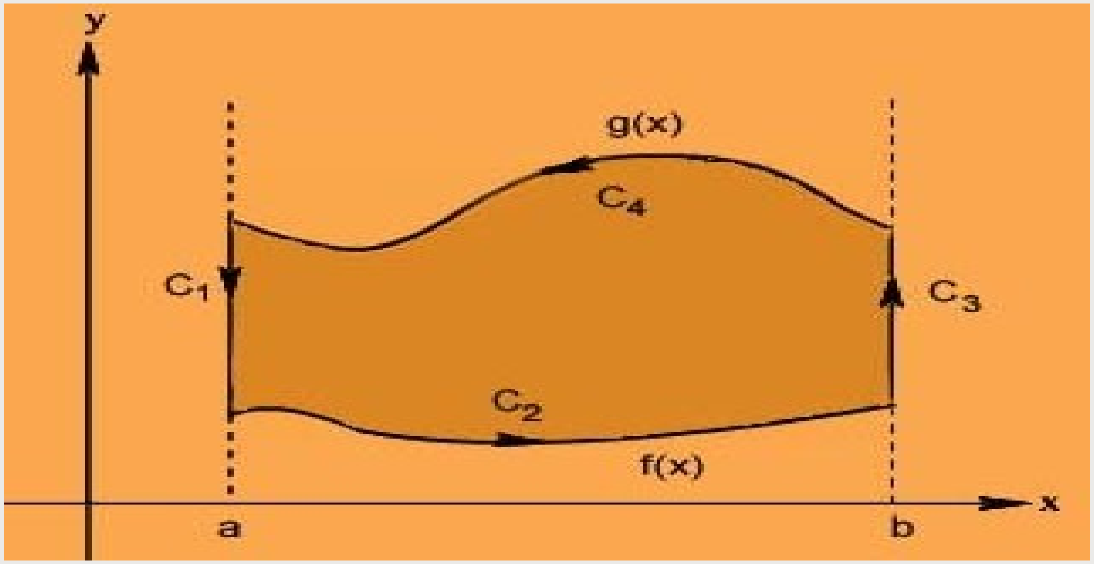
$$R := \left\{ (x,y) \in \mathbb{R}^2 \,|\, a \le x \le b, f(x) \le y \le g(x) \right\},$$

ere, f(x) &g(x) are continuously differentiable functions. Such a region is called

tically simple region or a type- I region in $\mathbb{R}^2$. Let denote the boundary of this

nain, being traversed in the counter-clockwise direction (i.e.,as it move along the

ndary, the domain lies to the left). Then the boundary can be broken into parts



$$C = C_1 \cup C_2 \cup C_3 \cup C_4 ,$$

Where C_1 And $C_2$are the vertical line segments,C_3 is the graph

$$\{(x, f(x)) \mid a \le x \le b\}$$

And$C_4$is the graph

$$\{(x, g(x)) \mid b \le x \le a\}.$$

Theorem (Green's theorem for simple domains)

e P, Q:U →IR continuously-differentiable scalar fields, $U \subset \mathbb{R}^2$

R be a vertically. Where, simple region in IR which can be represented as

$$R = \{(x,y) \mid a \le x \le b, f(x) \le y \le g(x)\}$$

, both the function f ,g are continuously differentiable. Further, Let U be such tha

boundary C are inside u . Then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

where, C is given the counter-clockwise orientation

oof

us write

$$C = C_1 \cup C_2 \cup C_3 \cup C_4,$$

re the curves C_1, C_2, C_3 and C_4 have, respectively, the following parameterizations

$$\begin{aligned}
&C_1 \text{ is } \mathbf{r}_1(y) := a\,\mathbf{i} + y\,\mathbf{j},\ g(a) \le y \le f(a),\\
&C_2 \text{ is } \mathbf{r}_2(x) := x\,\mathbf{i} + f(x)\,\mathbf{j},\ a \le x \le b,\\
&C_3 \text{ is } \mathbf{r}_3(y) := b\,\mathbf{i} + y\,\mathbf{j},\ f(b) \le y \le g(b),\\
&C_4 \text{ is } \mathbf{r}_4(x) := x\,\mathbf{i} + g(x)\,\mathbf{j},\ b \le x \le a,
\end{aligned}$$

,

$$\oint_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx = 0 + \int_{x=a}^{x=b} P(x, f(x))\,dx + 0 + \int_{x=b}^{x=a} P(x, g(x))\,dx$$

$$= \int_{x=a}^{x=b} \left[P(x, f(x)) - P(x, y(x)) \right] dx$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{f(x)} \frac{\partial P}{\partial y}(x, y)\, dy \right] dx$$

$$= -\iint_R \frac{\partial P}{\partial y}(x,y)\, dx\, dy$$

nilar calculation will give us,

$$\oint_C Q dy = \iint_R \frac{\partial Q}{\partial x}(x,y)\, dx\, dy$$

m (1) and (2) we get

$$\oint_C P dx + Q dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\, dy.$$

:

guments similar to the above theorem will tell us that conclusion of Green's theo
holds for regions R of the type :

$$R = \left\{ (x,y) \in \mathbb{R}^2 \mid C \le y \le d, f(y) \le x \le g(y) \right\},$$

$\mathbb{R}^2$.

Green's Theorem

is a simple closed curve enclosing a region *R* in the *xy-plane* and *P(x,y), Q(x,y)* and

er partial derivatives are continuous in *R*, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Where C is described in the anticlockwise direction.

mple: Verify Green's theorem in a plane for $\int_C \left[\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy\right]$

re C is the boundary Of the region defined by the lines $x=0,\ y=0$ and $x+y=1$.

ion

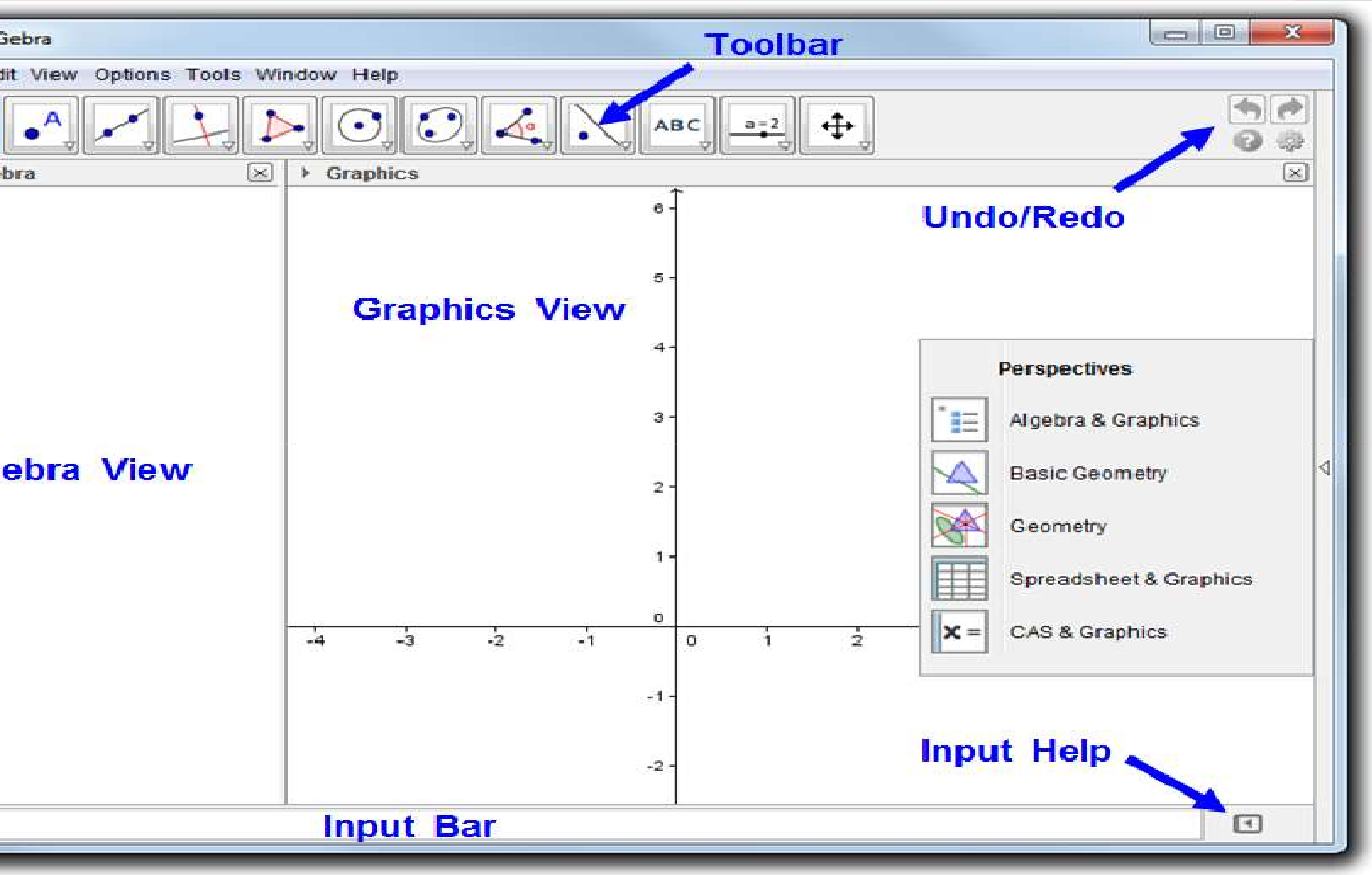
en's theorem is $\oint_C \left(P\,dx+Q\,dy\right)=\iint_R \left(Q_x-P_y\right)\,dx\,dy$

$$\int_C \left[\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy\right]=\iint_R 10y\,dx\,dy$$

$$L.S.\ of\ (1)=\int_{\substack{AB\\ \left(\substack{y=0\\ dy=0}\right)}}+\int_{\substack{BC\\ \left(\substack{x+y=1\\ x=1-y\\ dx=-dy}\right)}}+\int_{\substack{CA\\ \left(\substack{x=0\\ dx=0}\right)}}\left[\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy\right]$$

$$=\int_0^1 3x^2dx+\int_0^1\left(11y^2+4y-3\right)dy-\int_0^1 4y\,dy$$

$$R.S.\ of\ (1)=\int_0^1\int_0^{1-y}10y\,dx\,dy \quad \mathbf{=5/3}$$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Toolbar
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ABC
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Graphics
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Graphics View
ebra View
Perspectives
Algebra & Graphics
Basic Geometry
Geometry
Spreadsheet & Graphics
CAS & Graphics
Input Help
Input Bar

ut

In Algebra View

$a = 3 * x^2 - 8 * y^2$

$b = 4 * y - 6 * x * y$

$f := x = 0$

$g := y = 0$

$h := x + y = 1$

Intersect(f, g)

Intersect(g, h)

Intersect(h, f)

$derivative(a, y)$

$derivative(b, x)$

$text$ = "Along AB + BC + CA"

$0 < x < 1 - y$

$0 < y < 1$

CAS View

$$p = d - c$$

$$k := Integral(Integral(p(y), x, 0, 1 - y), y, 0, 1)$$

$$e := Integral(3 * x \wedge 2, x, 0, 1)$$

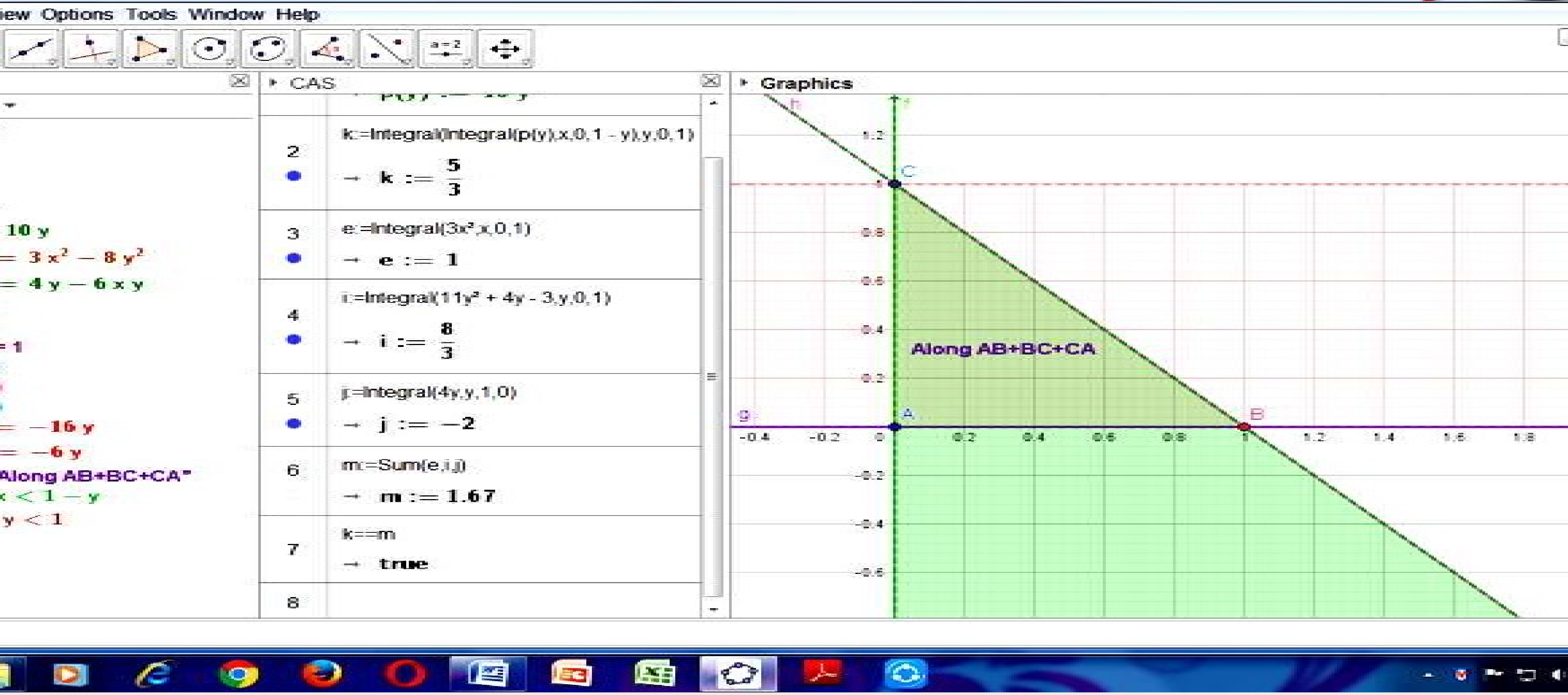
$$i := Integral(11 * y \wedge 2 + 4 * y - 3, y, 0, 1)$$

$$j := Integral(4 * y, y, 1, 0)$$

$$m := Sum(e, i, j)$$

$$k == m$$

hical Representation using ICT tool open source Software Geogebra



Answer: L.S. = R.S. = $\frac{5}{3}$

Green's Theorem is verified.

ce Exercises

erify Green's theorem in each of the following cases

$$P(x,y) = -xy^2; Q(x,y) = x^2 y;\ R\{(x,y) \mid x \ge 0, 0 \le y \le 1 - x^2\}$$

$$P(x,y) = 2xy; Q(x,y) = e^x + x^2;$$

re, R is the region inside the triangle with vertices $(0,0),(1,0),(1,1)$.

how that Green's theorem is applicable for the region R , the inside of the ellipse

$= 1$ and out side the circle $x^2 + y^2 = 1$ and hence compute

$dx + (x^2 + 2)dy$ where C is the boundary of the region.

cation of Green's Theorem:

aluations of line-integrals :

een's theorems help us to evaluate certain line integrals by evaluating the responding double integral. For example, we want to compute the line integra

$$I = \oint_C \left(5 - xy - y^2\right) dx - \left(2xy - x^2\right) dy$$

here C is the boundary of the square

$$R = \left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\right\}$$

g Green's theorem, we can convert this to a double integrals. Let C boundary of the region R, be given anticlockwise orientation.

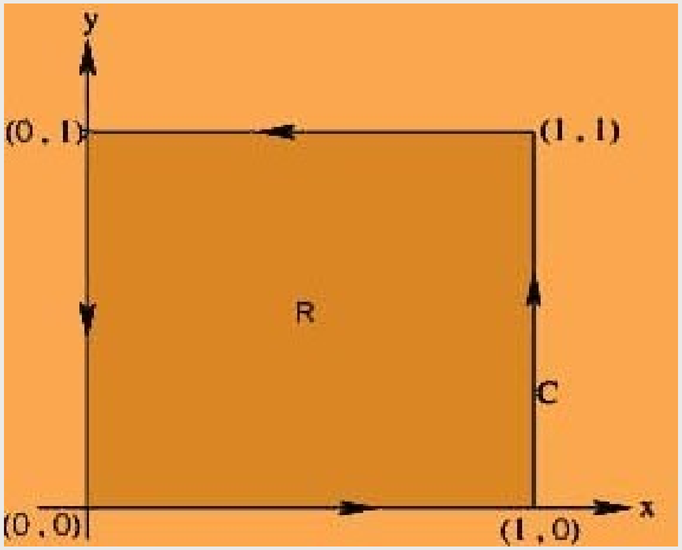


Figure: Design R, with oriented boundary

By Green's theorem,

$$I = \iint_R \left[\frac{\partial}{\partial x}(-2xy + x^2) - \frac{\partial}{\partial y}(5 - xy - y^2) \right] dxdy$$

$$= \int_0^1 \left(\int_0^1 (-2y + 2x + x + 2y) \right) dxdy$$

$$= \int_0^1 \left(\int_0^1 3x\, dx \right) dy = \frac{3}{2} \int_0^1 dy = \frac{3}{2}$$

eferences:

Books for Reference

dvanced Calculus
extbook by Lynn Harold Loomis and Shlomo Sternberg

dvanced Precalculus
aniel Kim, Dr. Michael Abramson

Advanced Calculus
Avner Friedman

wo and Three Dimensional Calculus
with Applications in Science and Engineering
Phil Dyke

YouTube links

- https://www.youtube.com/watch?v=8SwKD
- https://www.youtube.com/watch?v=kdTxN4
- https://www.youtube.com/watch?v=l5zJvZ
- https://www.youtube.com/watch?v=qdFD-0
- https://www.youtube.com/watch?v=TPov_

THANK YOU